RAMSEY GAME NUMBERS

WILLIAM GASARCH, CLYDE KRUSKAL, DAO LU, ERIC WEAVER, AND NATALIE WILKINSON

Abstract. Let H be a graph and m a natural number. Consider the following Game, which is inspired by Ramsey Theory: Player I (**blue**) and Player II (**red**) alternate coloring edges of K_m with their color. The first player to obtain a monochromatic H in his color wins. The Ramsey Game Number of H (denoted $RG(H)$) is the least m such that Player I has a winning strategy. We prove (1) if $n \ge 5$ then $RG(P_n) = n$ where P_n is the path on n vertices, (2) if $n \le 5$ then $RG(S_n) = 2n - 3$ where $S_n = K_{n-1,1}$, the star graph on n vertices, and (3) if $n \ge 6$ then $RG(S_n) \le 2n-4$. A computer program was used to simulate these Games and obtain these results; however, the final game-playing strategies involved in the proofs are humanly comprehensible.

1. Introduction

The following is a well known case of Ramsey's Theorem [**?**, **?**, **?**, **?**]: for all 2-colorings of the edges of K_6 there is a monochromatic K_3. We denote this statement by $R(K_3) = 6$. One can view this as a theorem about the following game.

Definition 1.1. *Let H be a graph and $m \in N$. The Ramsey Game for (H, m) is the following.*

*(1) There are two players **blue** and **red**. We assume Player I is **blue** and Player II is **red**.*
(2) They alternate coloring uncolored edges of K_m.
(3) The first one to obtain an H in their color wins.
(4) If all edges are colored and there is no monochromatic H, then the game is a tie.

Since $R(K_3) = 6$ the Ramsey Game for (K_3, 6) cannot be a tie, no matter what the players do (even if they play badly). The question arises as to what happens if the players play perfectly. The following theorem can be proved in a straightforward but tedious manner:

Theorem 1.1. *If both players play perfectly, then*

(1) The Ramsey Game for (K_3, 4) is a tie.
(2) The Ramsey Game for (K_3, 5) is a win for Player I.

In this paper we look at other theorems in Ramsey Theory and the corresponding games. The Ramsey Games we defined above, and a different type where the goal is to avoid getting the graph H in your own color, have been studied before, although mostly for the case of $H = K_n$. See [**?**] and the references therein.

Definition 1.2. *Let H be a graph.*

(1) $R(H)$ is the least m such that for all 2-colorings of the edges of K_m there is a monochromatic H. Such an m exists by Ramsey's Theorem.
(2) $RG(H)$ is the least m such that if both players play the Ramsey Game for (H, m) perfectly then Player I wins (one can show by a strategy stealing argumeng that if some player wins then it must be Player I). Such an m exists by Ramsey's Theorem.

Note that, for all graphs H, $RG(H) \le R(H)$.
We define several graphs H.

Definition 1.3. *Let $n \in \mathbb{N}$.*

(1) P_n is the path graph on n vertices.

(2) $S_n = K_{n-1,1}$ is the star graph on n vertices.
(3) K_n is the complete graph on n vertices.

The following are known (see [?] for references):
(1) For all $n \geq 2$, $R(P_n) = n + \lfloor \frac{n}{2} \rfloor - 1$.
(2) For all even $n \geq 4$, $R(S_n) = 2n - 1$
(3) For all odd $n \geq 3$, $R(S_n) = 2n$.

We show the following.
(1) For all $n \geq 5$, $RG(P_n) = n$.
(2)
$$\begin{cases} RG(S_n) = 2n - 3 & \text{if } 3 \leq n \leq 5 \\ RG(S_n) \leq 2n - 4 & \text{if } n \geq 6 \end{cases}$$

We obtained these results with the help of a program we wrote that used AI techniques (Minimax, Alpha-Beta pruning, and Monte-Carlo simulations) to play these games. Observing when Player I forced a win and when the game was a tie led to our conjectures. Observing how the program actually played inspired our proofs. For information on the techniques we used to write the program, and the code itself, see our website `http://ramseygames.nichesite.org/`.

2. Ramsey Games for Path Graphs

We used a minimax algorithm, which plays perfectly, to determine $RG(P_n)$ for $n = 3, 4, \ldots, 10$. Here are the results:

n	3	4	5	6	7	8	9
$RG(P_n)$	3	5	5	6	7	8	9

This motivated the conjecture that, for all $n \geq 5$, $RG(P_n) = n$, which we prove. For $n \geq 10$ we were unable to use the minimax algorithm; however, observing how a Monte Carlo algorithm played the game inspired or proof.

Definition 2.1. *A* blue path *is a list of distinct vertices $v_1, v_2, \ldots, v_k$ such that (v_i, v_{i+1}) is a blue edge (for $1 \leq i < k$), and v_1 and v_k have no other incident blue edges. The vertices v_1 and v_k are called* external vertices*; the other vertices on the path are* internal vertices*. The* size *of the path is k.*

Definition 2.2. *A* blue-isolated vertex *is a vertex with no incident blue edges.*

Definition 2.3. *The* red degree of a vertex *is the number of incident red edges.*

Definition 2.4. *The* total red degree of blue-isolated vertices *is the sum of the red degrees of all of the blue-isolated vertices.*

We can think of Player I's strategy as starting with n distinct blue paths of length 0, which are the n vertices, and at each turn connecting two blue paths together until after $n-1$ turns all of the vertices are connected into a single blue path. Player I does this in essentially two phases: In the first phase Player I connects pairs of blue-isolated vertices together forming one-edge blue paths. The phase ends when there is no longer a pair of blue-isolated vertices that can be connected. In the second phase, Player I connects the end points of pairs of blue paths merging two blue paths into a single blue path. The phase ends when there is a single blue path consisting of all n vertices. There is a small transition phase between Phases I and II, where remaining blue-isolated vertices are connected to blue edges and Player II makes a last move (so Phase II starts on Player I's turn).

Player I maintains the most flexibility of which two blue-isolated vertices to connect if the total red degree of blue-isolated vertices is small. Therefore, Player I makes moves that connect blue-isolated vertices with high red degree.

Phase I of Player I's strategy: At each turn, while there is at least one pair of blue-isolated vertices not connected by a red edge do the following: Find a blue-isolated vertex with maximum red degree. Connect it to another blue-isolated vertex that it is not already connected to (by a red edge). If there is more than one such vertex, pick one with maximum red degree. If there is a tie then pick one arbitrarily.

Lemma 2.1. *During Phase I, after each of Player I's turns the total red degree of blue-isolated vertices is either 0 or 1.*

Proof. We prove this by induction on the number of moves made.

Base case: Initially, Player I connects some two vertices (with a blue edge). The total red degree of blue-isolated vertices is 0.

Induction Hypothesis: Assume that after some turn for Player I, the total red degree of blue-isolated vertices is either 0 or 1.

Induction Step: Player II, can either connect two non-blue-isolated vertices, connect a blue-isolated vertex with a non-blue-isolated vertex, or connect two blue-isolated vertices.

Assume Player II connects two non-blue-isolated vertices. The total red degree of blue-isolated vertices remains the same. If it is 0, Player I connects any two blue-isolated vertices. If it is 1, Player I connects the blue-isolated vertex of red degree 1 to any other blue-isolated vertex. In either case, the total red degree of blue-isolated vertices is now 0. If there are no longer at least two blue-isolated vertices Phase I ends.

Assume Player II connects a blue-isolated vertex with a non-blue-isolated vertex. The total red degree of blue-isolated vertices increases by 1. If it is now 1, Player I connects the blue-isolated vertex to any other blue-isolated vertex, thereby reducing the total red degree of blue-isolated vertices to 0. If the total red degree of blue-isolated vertices is 2, then there are either two blue-isolated vertices that each have red degree 1, which Player I connects, or there is one blue-isolated vertex of degree 2, which Player I connects to any other blue-isolated vertex; in either case, the total red degree of blue-isolated vertices is reduced to 0. If there are no longer at least two blue-isolated vertices Phase I ends.

Assume Player II connects two blue-isolated vertices. If there are no other blue-isolated vertices Phase I ends. Otherwise, if one of the two end points of the edge now has red degree 2, Player I connects it to another blue-isolated vertex (which must have red degree 0). If neither of the two end points has red degree 2, Player I connects one end point to a blue-isolated vertex of red degree 1, if there is such a vertex; otherwise, Player I connects one end point to any blue-isolated vertex. In all of the cases, the total red degree of blue-isolated vertices is reduced to 1. If there are no longer at least two blue-isolated vertices Phase I ends. □

There are several cases for the end of Phase I: (1) There are no blue-isolated vertices. (2) There is one blue-isolated vertex. (3) There are two blue-isolated vertices connected by a red edge.

We start with Case (1). The other two cases are very similar.

Case 1: Assume that, at the end of Phase I, there are no blue-isolated vertices. (Note that the graph must have an even number of vertices, though we will not be using that.)

Definition 2.5. *A* blocking edge *is a red edge connecting two external vertices (of a blue path).*

BILL- CHANGE EXTERNAL VERTICES TO ENDPOINTS

BILL- SWAP RED AND BLUE

Player II makes one more move and then Phase II starts at turn $n/2+1$. Both Players I and II have had $n/2$ turns, there are $n/2$ blue paths (of length one), every vertex is an external vertex, and there are $n/2$ blocking edges.

Phase 2 of Player I's strategy: At each turn, while there is more than one blue path do the following: Find an external vertex, say x, with maximum red degree. Connect it to an external vertex of another path (that it is not already connected to x by a red edge); if there is more than one such vertex, pick one with maximum red degree.

Lemma 2.2. *Let $n \geq 8$. Assume that, at the end of Phase I, there are no blue-isolated vertices. During Phase II, at (the beginning of) turn $n/2 + t$ there are $n/2 - (t-1)$ blue paths and at most $n/2 - (t-1)$ blocking edges.*

Proof. We prove this by induction on t.
Base Case: Initially, when $t = 1$, there are $n/2$ blue paths and $n/2$ blocking edges.
Induction Hypothesis: Assume that at turn $n/2 + (t-1)$ there are $n/2 - (t-2)$ blue paths and at most $n/2 - (t-2)$ blocking edges.

Induction Step: Find an external vertex with maximum red degree, say x. There are $n/2 - (t-2) - 1$ blue paths that it could connect to, so there are $n - 2t + 2$ potential external vertices to connect to. There are at most $n/2 - (t-2)$ blocking edges, which is fewer ($t < n/2$, because Player I can win in $n-1$ moves). Player I connects the x to the external vertex, say y, with maximum red degree, such that (x, y) is not already a red edge. If x has at least two incident blocking edges, then the number of blocking edges decreases by at least two. If x and y each have exactly one incident blocking edge, then the number of blocking edges decreases by exactly two. If x has exactly one incident blocking edge and y has none, there must be only one blocking edge, so the number of blocking edges decreases to zero. If x does not have a blocking edge then there are none at all.

Player II then connects two vertices, increasing the number of blocking edges by at most one. The result follows. □

Case 2: Assume that, at the end of Phase I, there is one blue-isolated vertex, so n is odd. In this case, Player II makes a move, Player I connects the blue-isolated vertex to some other vertex, and Player II makes another move. Phase II then starts. Both Players I and II have had $(n+1)/2$ turns, there are $(n-1)/2$ blue paths, and there are at most $(n+1)/2$ blocking edges.

The leads to the following lemma with the same proof as the previous lemma.

COMMENT FROM BILL- IN CASE 2 WHAT IS PLAYER I's STRATEGY IN PHASE 2?

Lemma 2.3. *Let $n \geq 5$. Assume that, at the end of Phase I, there is one blue-isolated vertex. During Phase II, at (the beginning of) turn $(n+1)/2 + t$ there are $(n-1)/2 - (t-1)$ blue paths and at most $(n+1)/2 - (t-1)$ blocking edges.*

Case 3: There are two blue-isolated vertices connected by a red edge, so n is even. One of the two vertices, say x, must have red degree one, and the other vertex, say y, must have red degree one or two. Assume that $n \geq 8$. ($n = 6$ is a special case.) In the figures that follow, green edges are blocking edges that have been removed.

- Case (a): Assume that y has red degree one. Then Player I connects x to vertex a that has maximum red degree. Because a must already be incident on a blue edge, any red edges incident on a are no longer blocking edges. If there is more than one red edge incident on a, then at least two blocking edges are no longer blocking edges. (See Fig. **??**.) Then Player II makes a move, and then Player I connects y to any vertex. Otherwise, there is exactly one red edge incident on a. Let (a, b) be the blue edge incident on a. Since $n > 6$, there must be some other blue edge, say (c, d) with a different incident red edge, say (c, e). If Player II takes edge (y, c) then Player I takes edge (y, d) so that (y, c) is no longer a blocking edge. (See Fig. **??**.) If Player II take any other edge Player I takes edge (y, c) and (c, e) is no longer a blocking edge. (See Fig. **??**.)

- Case (b): Assume that y has red degree two. Then there is another red edge (y, z). Then Player I connects x to z (See Fig. **??**), and since z is already connected to a blue edge, (y, z) is no longer a blocking edge. Player II has three choices of moves:

FIGURE 1. There is more than one red edge incident on a.

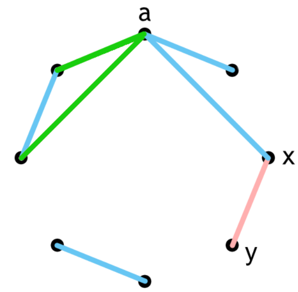
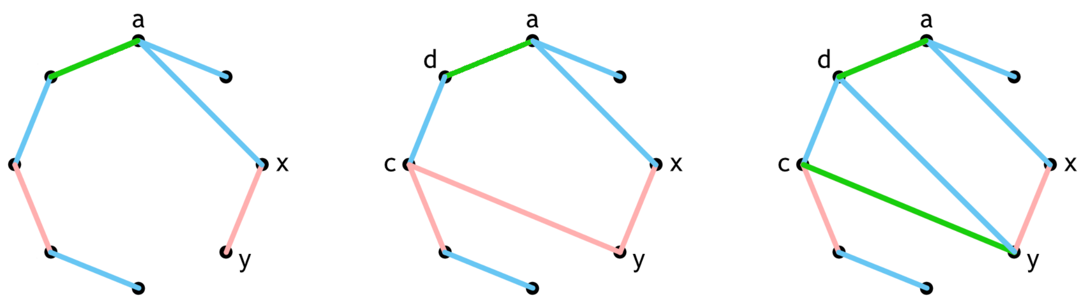
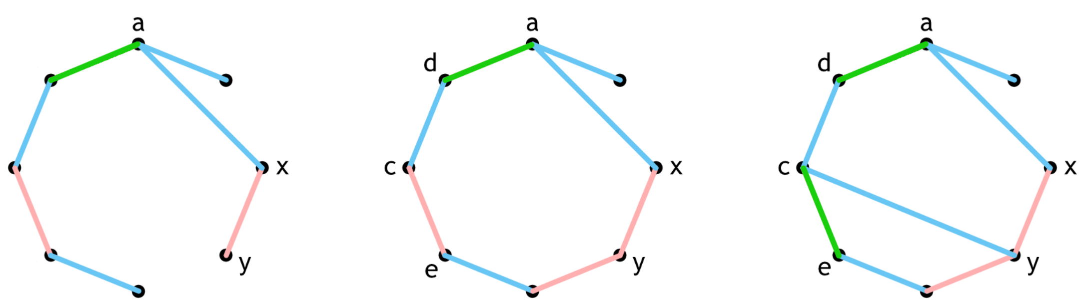


FIGURE 2. There is exactly one red edge incident on a.



(A) Player I takes edge (x, a). (B) Player II takes (y, c). (C) Player I takes edge (y, d).

FIGURE 4. There is exactly one red edge incident on a.



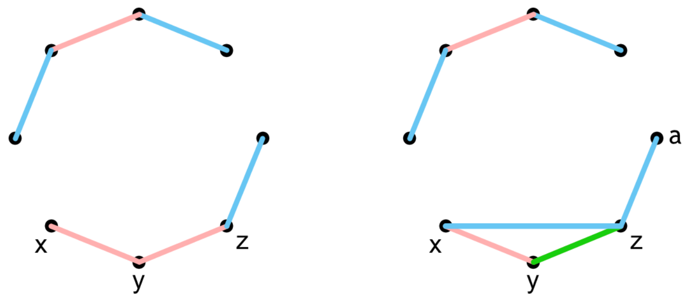
(A) Player I takes edge (x, a). (B) Player II takes any edge besides (y, c). (C) Player I takes edge (y, c).

i Take edge (x, a), where (z, a) is the blue edge incident on z. It is not a blocking edge. (See Fig. **??**.) Player I can take any edge incident on y.

ii Take edge (x, a), where (a, b) is some blue edge. Then Player I takes edge (y, a), so (x, a) is not a blocking edge. (See Fig. **??**.)

iii Take edge (y, a), where (a, b) is some blue edge. Then Player I takes edge (y, b), so (y, a) is not a blocking edge. (See Fig. **??**.)

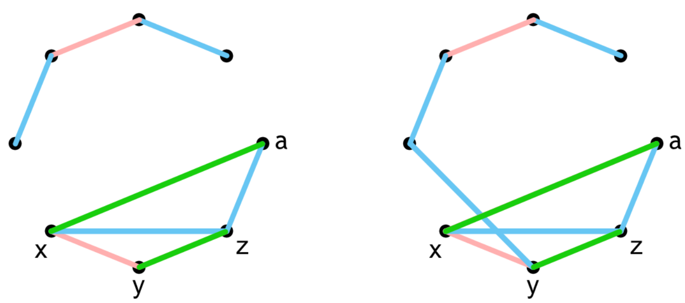
FIGURE 6. y has red degree two.



(A) y has red degree two.

(B) Player I connects (x, z).

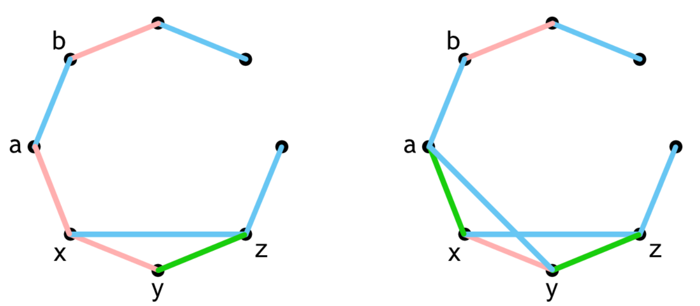
FIGURE 8



(A) Player II takes edge (x, a).

(B) Player I takes any edge incident on y.

FIGURE 10

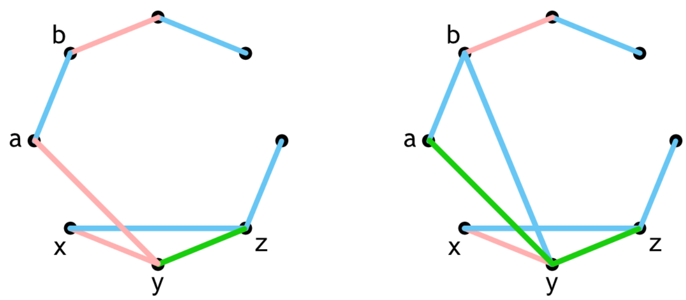


(A) Player II takes edge (x, a).

(B) Player I takes edge (y, a).

Player II makes one more move. In all cases, at least two red edges are not blocking edges. Both Players I and II have had $n/2+1$ turns, there are $n/2-1$ blue paths, and there are at most $n/2-1$ blocking edges. Phase II starts at turn $n/2+2$.

Figure 12



(a) Player II takes edge (y, a). (b) Player I takes edge (y, b).

The leads to the following lemma with the same proof as the previous two lemmas.

Lemma 2.4. *Let $n \geq 8$. Assume that, at the end of Phase I, there are two blue-isolated vertices connected by a red edge. During Phase II, at (the beginning of) turn $n/2 + 1 + t$ there are $n/2 - 1 - (t - 1)$ blue paths and at most $n/2 - t$ blocking edges.*

For all three of the cases stated above, at the start of turn $n - 1$, at most two of the four ways to connect the last two paths together have blocking edges so Player I has at least two ways to construct a path of size n and thus win. added "for all...above"

changed it up a bit

Theorem 2.5. *In Ramsey Games for Path Graphs of size $n \geq 5$, and where the goal is to create a path of length n, Player I wins in $n - 1$ turns.*

Corollary 2.6. *For $n \geq 5$, $RG(P_n) = n$.*

3. Ramsey Games for Star Graphs

We used a minimax algorithm, which plays perfectly, to determine $RG(S_n)$ for $n = 3, 4, 5$.

n	3	4	5
$RG(S_n)$	3	5	7

A Monte Carlo algorithm seemed to indicate that $RG(S_6) = 8$ and $RG(S_7) = 10$. The table above and the results of the Monto Carlo algorithm motivated a conjecture for an upper bound on $RG(S_n)$, which we prove. For large n we were unable to use the minimax algorithm; however, observing how the Monte-Carlo algorithm played the game inspired the game-playing strategies involved in our proof.

Recall that S_n has a center and $n - 1$ spokes. Again, we refer to Player 1 as "**blue**" and Player 2 as "**red**."

Theorem 3.1.

$$\begin{cases} RG(S_n) = 2n - 3 & \text{for } 3 \leq n \leq 5 \\ RG(S_n) \leq 2n - 4 & \text{for } n \geq 6 \end{cases}$$

Definition 3.1. *The* weight *of a vertex (relative to a player) is the number of incident edges of a player's color.*

Definition 3.2. *The* net weight *of a vertex (relative to a player) is the number of incident edges of a player's color, minus the number of incident edges of the opponent's color.*

In games where $3 \leq n \leq 5$ we say that a vertex with net weight at least 2 is a *win vertex*, and a vertex with net weight 1 is a *threat vertex*. In games where $n \geq 6$ we say that a vertex with net weight at least 2 is a *win vertex*, a vertex with net weight 1 is a *threat vertex*, and a vertex with net weight 1 is a *weak threat vertex*.

The term "threat" assumes that it is the opponent's turn. If a player has a threat vertex at the beginning of his own turn, it is a winning position (because it can be turned into a win vertex on his turn), unless the opponent has a vertex that has a greater weight than the threat vertex.

Lemma 3.2. *Player I wins the star game if at the beginning of his turn he has a threat vertex, and Player II has no vertex that has a greater weight than Player I's threat vertex.*

Proof. For games where $m = 2n - 3$, the win vertex Player I will create from the threat vertex at the beginning of the turn has net weight 2. If Player I uses this win vertex as the center of a star graph, he has already connected 3 of the vertices needed for the star graph. Thus, he needs $n - 3$ more degrees of his color incident on the center vertex. For the remaining vertices in the graph, Player I can connect at least half of them, since Player II can block him from connecting the other half. The game board has $2n - 3$ vertices, and of these, at least 3 are part of the star already, leaving $2n - 6$ vertices' connections to the center in question. But of these $2n - 6$, since Player I is able to connect to half of them, or $n - 3$, it is just enough to create a star of the winning size. For games where $m = 2n - 4$, the logic is similar. If a win vertex has net weight 3, then Player I knows 4 vertices on the winning star graph already. Then, he only needs $n - 4$ more vertices. But he has $2n - 8$ vertices left whose connections to the center are in question, and he can connect at least half, or $n - 4$, of them, which is exactly how many he needs to win.

As long as Player II does not have a vertex of greater weight than Player I's threat vertex (that will soon be a win vertex), Player I can continue to connect to his threat vertex until he wins, without worrying about blocking Player II's attempts to win. However, if Player II does have a vertex that has a greater weight, even if it does not have a greater net weight, he could potential create a threat vertex out of it, and then be closer to winning than Player I because Player II has more edges of his color incident on his threat vertex (so his star is closer to completion). Then Player I no longer has the advantage, but instead is forced to parry Player II's threats. □

We will show that for $3 \leq n \leq 5$ and $m \geq 2n - 3$ or $n \geq 6$ and $m \geq 2n - 4$, Player I wins. The strategy for the former is rather trivial, but for the latter, it is to create a threat vertex, and then parlay that threat into a double threat (two simultaneous threat vertices), which cannot both be parried. See Fig. **??** and Fig. **??** for two example games that demonstrate the strategy when $n = 6$ and $m = 8$.

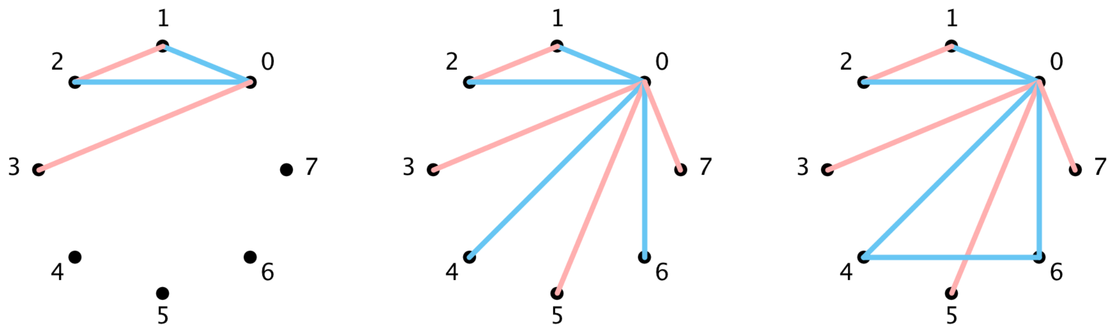
Lemma 3.3. *Let $3 \leq n \leq 5$ and $m \geq 2n - 3$. Player I wins the star game.*

Proof. On Player I's turn, he connects two vertices. Now he already has a double threat, and Player II cannot parry both. Thus Player I will have a threat vertex at the beginning of his next turn, and by Lemma **??**, he wins. □

Lemma 3.4. *Let $n \geq 6$ and $m \geq 2n - 4$. Player I will win if at some point in the game, which we call Z:*

(1) There exist a weak threat vertex (of Player I's) and at least 4 vertices not yet connected to the weak threat vertex, at least 3 of which are not connect to any vertex.
(2) At any vertex not yet connected to the weak threat vertex, Player 2's weight is no greater than 1 and the net weight is no greater than 0.
(3) Player II has no vertex with a greater weight than Player I's weak threat vertex, which has weight 2.
(4) It is Player I's turn.

FIGURE 14. Example Game I.



(A) On the first move, Player I must pick some edge, say $(0, 1)$. Nodes 0 and 1 are initially weak threats. Player II picks edge $(1, 2)$, say, so vertex 0 remains a weak threat (of Player I's), and vertex 1 is no longer a weak threat, and vertex 2 becomes a weak threat (of Player II's).

(B) Player I picks edge $(0, 4)$, making 0 a threat again. Player II is again forced to pick an edge incident on 0, say $(0, 5)$. Player I picks edge, say $(0, 6)$, making 0 a threat again. Player II must pick edge $(0, 7)$.

(C) Now Player I now picks edge $(4, 6)$, which creates a double threat: vertices 4 and 6. Whatever Player II does, either vertex 4 or 6 will be a threat at the beginning of Player I's next turn, which, by Lemma **??**, is a win.

Proof. On Player I's turn, he will connect the weak threat vertex to an unconnected vertex to continue the threat. On Player II's turn, he must also connect the threat vertex to any available vertex to parry the threat, or he will lose by Lemma **??**. Condition 2 of Lemma **??** insures that in the process of parrying Player I's threat vertex Player II cannot create a threat vertex or a vertex with a greater weight than Player 1's threat vertex. There are at least 3 unconnected vertices left at Z, so Player I can create weak threats on at least 2 of them (because it is Player I's turn first). Then, 4 moves after Z, it is Player I's turn again, and he can connect these 2 vertices, creating a double threat. Player II can only parry one of them, so that on Player I's next turn he still has a threat vertex remaining, and thus Player I wins by Lemma **??**. ☐

Lemma 3.5. *Let $n \geq 6$ and $m \geq 2n - 4$. Player I wins the star game.*

Proof. On move 1 player I picks some edge, say $(0, 1)$. Player II has two possible responses: pick an edge incident on either 0 or 1, or pick some independent edge.

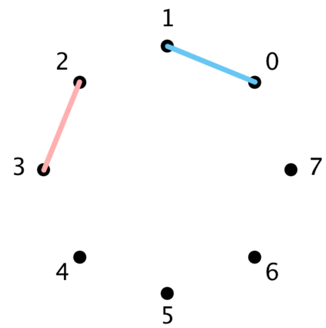
Case 1: (See Fig. **??**) Player II picks an edge incident on either 0 or 1, say $(1, 2)$. Then Player I chooses $(0, 2)$. If Player II does not connect to vertex 0 on the next turn, by Lemma **??** Player I wins. If Player II does connect to vertex 0 and any vertex, then on Player I's turn the conditions of Lemma **??** will be fulfilled, and by that lemma Player I will win.

Case 2: Player II picks some independent edge, say $(2, 3)$. Player I picks edge $(0, 2)$. Player II must choose an edge incident on vertex 0 because it is a threat. There are two cases:

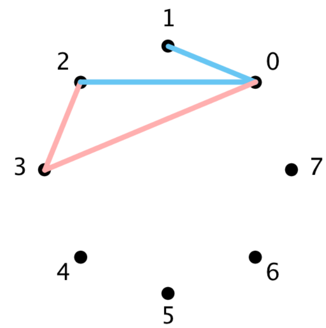
Case 2a: (See Fig. **??**) Player II picks edge $(0, 3)$, so that he also creates a threat. Player I picks edge $(1, 3)$ (transferring the threat to vertex 1 and parrying Player II's threat). Player II either connects vertex 1 to 2 or connects vertex 1 to an unconnected vertex. Either way, the conditions of Lemma **??** are fulfilled, so by that lemma Player I will win.

Case 2b: Player II picks an edge incident on a blue-isolated vertex, say $(0, 4)$. Player I picks edge $(0, 3)$. Now the only vertices left unconnected to 0 are those unconnected to any vertex, and at least three of them remain. Player II connects one of these to vertex 0, Player I does the same, and Player II must connect another to vertex 0 to parry the threat. At this point, it is Player I's

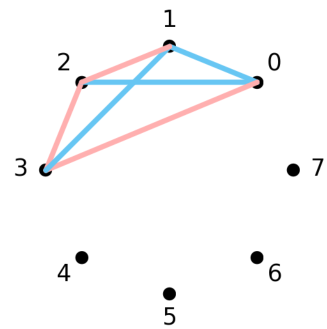
FIGURE 16. Example Game 2.



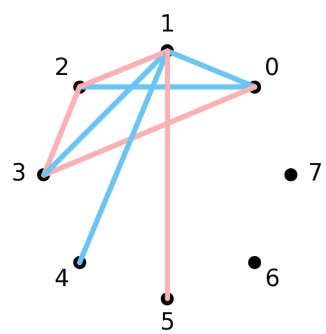
(A) On move 1 player I must pick some edge, say $(0, 1)$. Player II picks some independent edge, say $(2, 3)$.



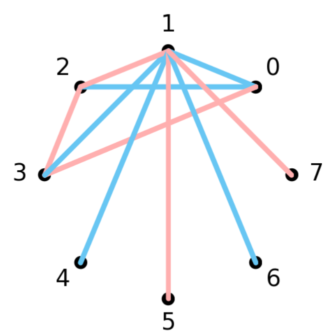
(B) Player I picks edge $(0, 2)$, so that vertex 0 is a threat, and vertices 2 and 3 are not both weak threats for Player 2. Player II picks edge $(0, 3)$ to stop the threat at vertex 0 and to create his own threat at vertex 3.



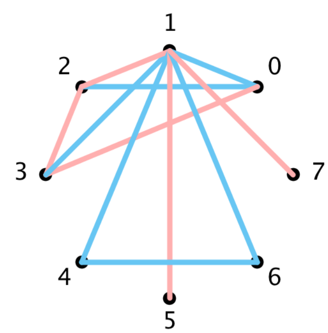
(C) Player I picks edge $(1, 3)$, which transfers the threat from vertex 0 to vertex 1, and parries Player II's threat at vertex 3. Player II must pick an edge incident on 1, say $(1, 2)$.



(D) Player I picks an edge incident on 1, say $(1, 4)$. Once again, Player II must pick an edge incident on 1, say $(1, 5)$.



(E) Player I picks another edge incident on 1, say $(1, 6)$. Player II must pick edge $(1, 7)$.



(F) Now Player I now picks edge $(4, 6)$, which creates a double threat: vertices 4 and 6. Whatever Player II does, either vertex 4 or 6 will be a threat at the beginning of Player I's next turn, which, by Lemma **??**, is a win.

turn, and Player I has at least 2 vertices that are weak threats incident to only one edge (his own). One of these vertices was connected the previous turn, and the other is vertex 1. Player 1 connects these to create a double threat, and on Player II's turn he can only parry one. Moreover, there is no vertex with net weight or weight (of Player II's) greater than 1, so Lemma **??** holds, and by it Player I wins. □

We can now prove Theorem **??**, restated below for reference:

$$\begin{cases} RG(S_n) = 2n - 3 & \text{if } 3 \leq n \leq 5 \\ RG(S_n) \leq 2n - 4 & \text{if } n \geq 6 \end{cases}$$

Proof. For $3 \leq n \leq 5$, the result follows from Lemma **??** and for $n \geq 6$, the result follows from Lemma **??**. □

4. Results and Open Problems

(1) We have completely solved the Ramsey Game Problem for Path Graphs.
(2) For $n \geq 6$ we have the following upper bound for playing the Ramsey Game for Star Graphs: $RG(S_n) \leq 2n - 4$. Our experience playing the game indicates the bound is tight, which we conjecture.
(3) We used a minimax algorithm for playing the Ramsey Game for Cycle Graphs (C_n is the graph that is a cycle on n vertices). We obtained the following holds:

n	3	4	5	6
$RG(C_n)$	5	6	6	7

We conjecture, admittedly on very little data, that for $n \geq 5$, $RG(C_n) = n + 1$.
(4) One can study the Ramsey Game for Wheel Graphs, Grid Graphs, Complete Graphs, and many other graphs such as those in the paper on Small Ramsey Numbers [**?**]. One can show that $RG(K_3) = 5$; however $RG(K_4)$ seems difficult.

One can look at the following variants of Ramsey Games:

(1) Maker-Breaker games: Player I is trying to *make* a graph in his color, whereas Player II is trying to *prevent* this from happening. The Ramsey Game for P_n is equivalent to the Maker-Breaker version. We have reason to believe that the Ramsey Game for S_n is not equivalent to the Maker-Breaker version. For all others we do not know. This seems like a promising research area. Asymptotic results on maker-breaker games have been studied extensively [**?**].
(2) Ramsey Avoidance games: Instead of trying to *obtain* a graph in his color, a player might wish to *avoid* a graph in his color. The case of avoiding K_3 is called *SIM* and has been studied extensively. Other cases have also been studied. See [**?**] and the references therein.

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University of Maryland at College Park, Department of Computer Science, College Park, MD, 20741

University of Maryland at College Park, Department of Computer Science, College Park, MD, 20741

Princeton University, Department of Computer Science, Princeton, NJ 08540, USA

Grove City College, Department of Computer Science, Grove City, PA 16127, USA

Roanoke College, Department of Mathematics, Salem, VA 24153, USA