Andy Drucker’s Derivation of Can Ramsey from Sz Theorem

Recall Sz’s theorem:

**Def 0.1** Let $A \subseteq \mathbb{N}$. The *density of* $A$ is

$$\limsup_{n \to \infty} \frac{|A \cup \{1, \ldots, n\}|}{n}.$$ 

**Theorem 0.2** Let $\epsilon \in \mathbb{R}^+$ and $k \in \mathbb{N}$. If $A \subseteq \mathbb{N}$ of density $\epsilon$ then $A$ contains a $k$-AP.

Here is a finite version you can get from compactness:

**Theorem 0.3** Let $\epsilon \in \mathbb{R}^+$ and $k \in \mathbb{N}$. There exists $N = N(i, \epsilon)$ such that the following holds: if $A \subseteq [N]$ of density $\epsilon$ then $A$ contains a $k$-AP.

We want to prove the Can Ramsey Theorem:

**Theorem 0.4** Let $k \in \mathbb{N}$. There exists $N = N(k)$ such that any coloring of $[N]$ has either a mono $k$-AP or a rainbow $k$-AP.

**Proof:**

We will set $\epsilon = \frac{1}{2^k}$. Let $N = N(k, \epsilon)$ from Theorem 0.3.

Let $\text{COL} : [N] \to \omega$. If some color has density $\geq \frac{1}{2^k}$ then by Theorem 0.3 there is a mono $k$-AP. Hence we assume all colors have density $< \frac{1}{2^k}$.

Consider the following prob experiment: pick $(a, d) \in [N] \times [N]$ and then look at the set

$$a \pmod{N}, a+d \pmod{N}, a+2d \pmod{N}, \ldots, a+(k-1)d \pmod{N}$$

and return Y if these are all diff colors, and N if there are two that are the same color.

What is the prob that two of the $a + dj$ and $a + dj'$ have the same color? \[ \]