



Universal Night episode 7 **ASTRO OLYMPIAD**



Cosmology

90 minutes

Contestant name:.....

School name:.....

Start time:.....

End time:.....

Note: The following equations might be useful:

Friedman

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{\kappa c^2}{R_0^2 a^2} + \frac{\Lambda}{3} = H^2 \quad (1)$$

Fluid

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0 \quad (2)$$

Acceleration

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) + \frac{\Lambda}{3} \quad (3)$$

Equation of state

$$P = \omega\epsilon \quad (4)$$

Unless stated otherwise take $H_0 = 72 \text{ km s}^{-1} \text{ Mpc}^{-1}$.

- 1) In a cosmological model, the radius R of the universe is a function of the age t of the universe. The function R satisfies the three conditions:

$$R(0) = 0, \quad R'(t) > 0 \text{ for } t > 0, \quad R''(t) < 0 \text{ for } t > 0, \quad (*)$$

where R'' denotes the second derivative of R . The function H is defined by

$$H(t) = \frac{R'(t)}{R(t)}$$

i. Sketch a graph of $R(t)$. By considering a tangent to the graph, show that $t < 1/H(t)$.

ii. Observations reveal that $H(t) = a/t$, where a is constant. Derive an expression for $R(t)$.

What range of values of a is consistent with the three conditions $(*)$?

iii. Suppose, instead, that observations reveal that $H(t) = bt^{-2}$, where b is constant. Show that this is not consistent with conditions $(*)$ for any value of b .

- 2) Explain the difference between Einstein universe and the De Sitter universe.

- i. Consider Einstein's static universe, in which the attractive force of the matter density ρ is exactly balanced by the repulsive force of the cosmological constant, $\Lambda = 4\pi G\rho$.

Suppose that some of the matter is converted into radiation (by stars, for instance). Will the universe start to expand or contract? Explain your answer.

- ii. If $\rho = 3 \times 10^{-27} \text{ kg m}^{-3}$, what is the radius of curvature R_0 of Einstein's static universe? How long would it take a photon to circumnavigate such a universe?

By considering both the Friedman and acceleration equations, and assuming a pressure-less Universe, demonstrate that in order to have a static Universe we must have a closed Universe with a positive vacuum energy. Using either physical arguments or mathematics, demonstrate that this solution must be unstable.

3)

- 1. Give a comparative description on Standard Big Bang model and the Λ CDM model of the universe.
- 2. Draw the graphs of $R = R(t)$ for Friedman models,
 - first with $\Lambda = 0$: (a) $K > 0$, (b) $K = 0$, (c) $K < 1$, and then (d) with $\Lambda > 0$.
- 3. Is it possible for a closed Universe to evolve to become an open Universe? Give a reason for your answer.
- 4. Draw the space time diagram and a conformal diagram for a Black hole formed by collapsing matter.
- 5. Identify and explain in brief the major eras of big bang cosmology.

4)

1. Consider an energy component of the universe with equation of state $p = \omega\rho$ ($c = 1$ units). How does its energy density ρ change with expansion factor $a(t)$?
2. Given that Friedman equation can be written as

$$1 - \Omega(t) = -\frac{kc^2}{R_0^2 a(t) H(t)^2}$$

Show how the density of the universe $1 - \Omega(t)$ evolves for universes dominated by matter, radiation and cosmological constant.

3. a) Outline the evidence for dark matter in galaxies and clusters.
- b) Show that the rotation curve of a galaxy should follow the form

$$v_{orb} = \sqrt{\frac{GM}{R}}$$

c) An alternative to dark matter is that the law of gravity should be modified.

If $F = GMm^\gamma/R$, what γ would be required to remove the requirement for dark matter

5)

1. This problem indicates the origin of the equation of state $p = \rho c^2/3$ for radiation. An ideal gas has pressure

$$p = \frac{1}{3} n \langle \mathbf{v} \cdot \mathbf{p} \rangle,$$

where $\langle \dots \rangle$ indicates an average over the direction of particle motions. Here n is for the number density, p is for pressure and $\mathbf{p}$ for momentum. Using the relationship between the photon energy and momentum, show that this gives

$$p = \frac{1}{3} n \langle E \rangle,$$

where $\langle E \rangle$, is the mean photon energy.

Hence demonstrate the equation of state for radiation.

2. Use the acceleration equation and the definition of critical density to show that a radiation-dominated Universe has $q_0 = \Omega_0$.
3. Identify a sufficient and necessary condition that must be satisfied by the equation of state if q_0 is to be negative.