EM 509: Stochastic Processes

Class Notes

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Introduction: Definition

- A random process or stochastic process is a family of random variables indexed by an ordered or index set denoted by T. The index set can be either discrete or continuous and usually it refers to time.

- This could refer to a finite family of random variables, but in practice the term usually refers to infinite families.

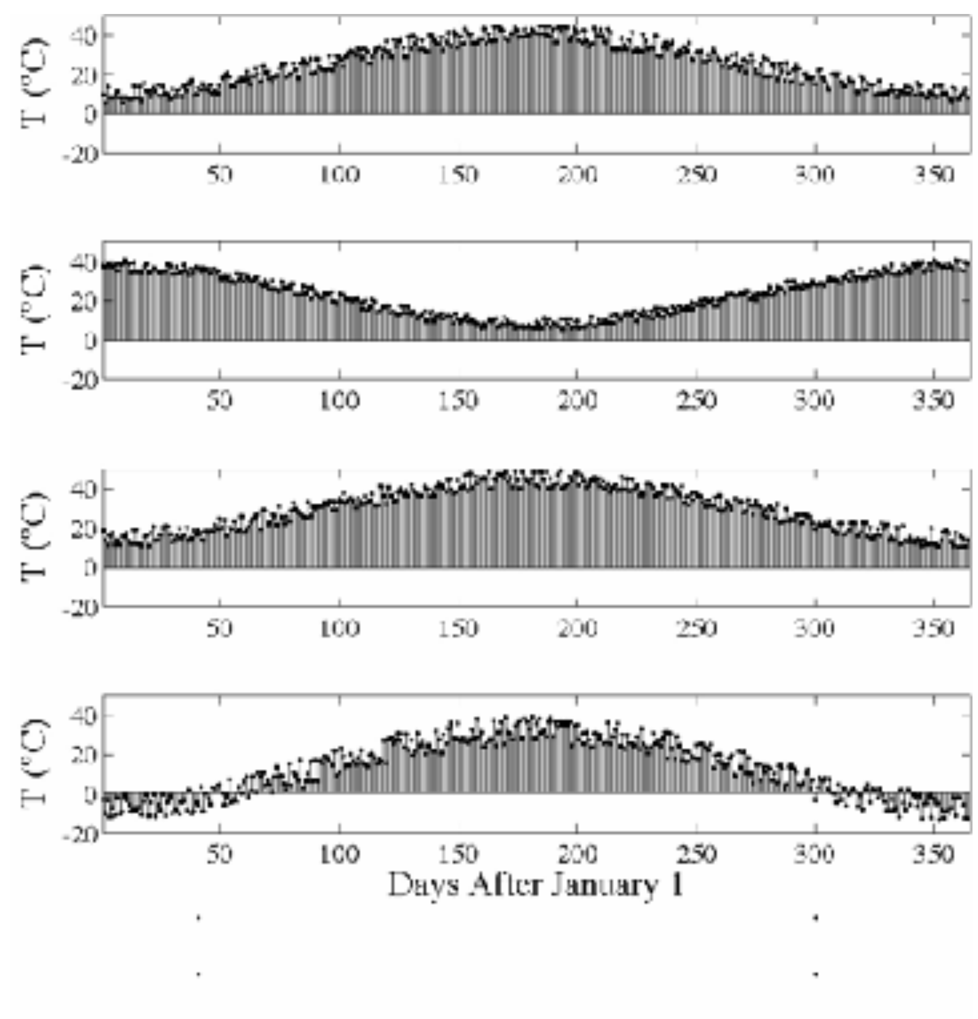
- The need for working with infinite families of random variables arises when there are indeterminate amount of data to model.

- Recall a random variable completely defines the possible events in the underlying probability space. Therefore in the study of stochastic processes all we need is the codomain for the random variable which is called the state space of the process.

- A stochastic process is denoted by X_t **or** $X(t)$ **for** $t \in T$. When T is discrete the notation X_n is also used.

Introduction: Examples

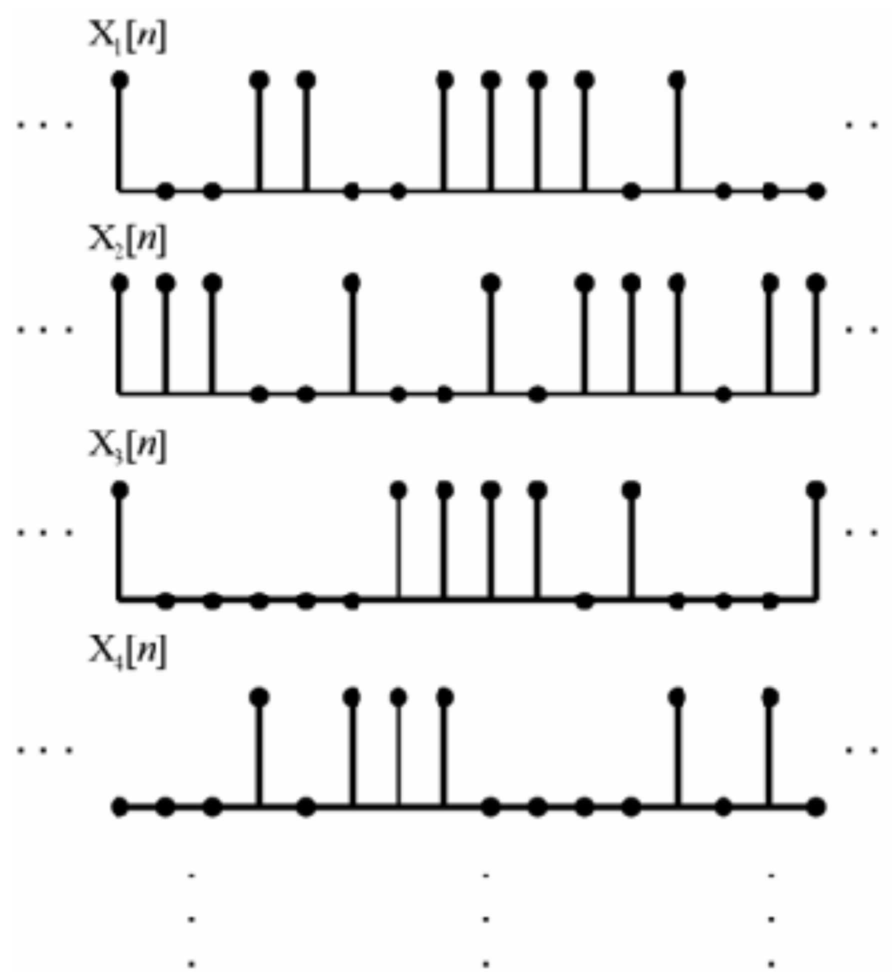
- Temperature:
- Suppose we place a temperature sensor at every airport control tower in the world and record the temperature at noon every day for a year.
- This is a discrete-time stochastic process with continuous state space.



Introduction: Examples

- Flipping A Coin:

Suppose there is a large number of people, each flipping a fair coin every minute. If we assign the value 1 to a head and the value 0 to a tail we have a discrete time stochastic process with discrete state space.



Introduction: Examples

- A Random walk:

- A random walk is a fundamental physical process in which a random walker; a particle, an atom, a measurement, an individual moves in random steps. A typical example is the spreading of an atom throughout a fluid.

- For example in one dimension, let X_n be a random variable denoting the position at time n of a moving particle for n=0,1,2, …...

- The particle will move around the integer {…,-2,-1,0,1,2,….}. For every single point of time, there is a jump of one step for the particle with some probability say for example 1/2(a jump could be to left or right).

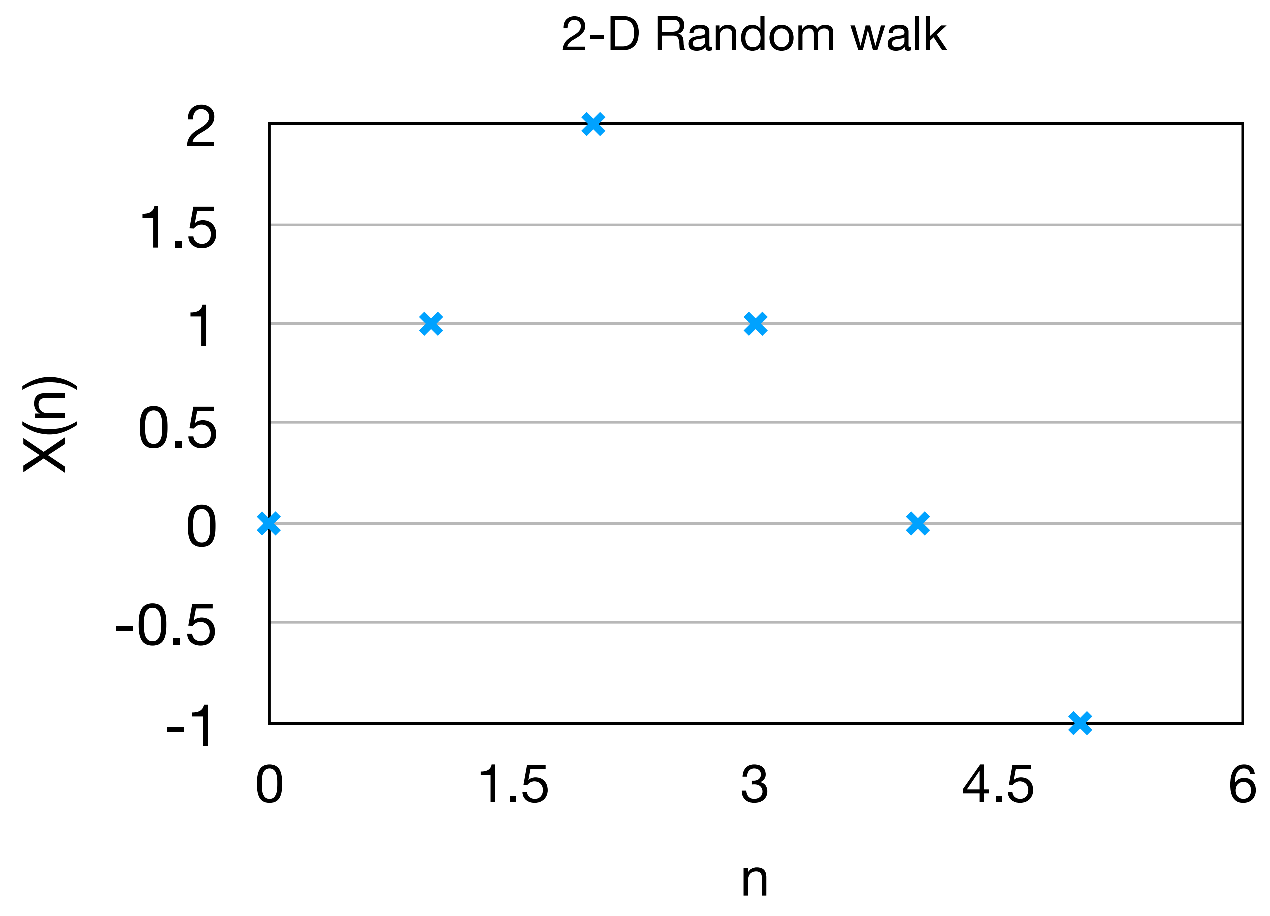
- The jumps at time n=1,2,3,… are independent and the process is called simple random walk. This is a discrete time stochastic process with discrete state space.

Introduction: Examples

Random walk cntd. In general,

$$X_n = X_{n-1} + Z_n, \;\; Z_n = 1, -1; \;\; P(Z_n = 1) = P(Z_n = -1) = \frac{1}{2};$$

$$X_n = X_0 + Z_1 + Z_2 + \cdots + Z_n, \;\; X_0 = 0$$



Introduction: Examples

- **Stock Market:**

- The price of a particular stock counter listed on the stock exchange as a function of time is a stochastic process. Here the state space is the set of all prices of that particular counter throughout the day. This is a continuous time stochastic process with continuous state space.

- **n-th Bit of a binary expansion:**

- Let X_n denotes the n-th bit in the binary expansion of a number in $[0,1)$. For example, $X_1^{-1}(0) = [0,0.5)$ and $X_1^{-1}(1) = [0.5,1)$. This is a discrete-time random process with discrete state space.

- **Board Game:**

- A game which moves are determined by dice such as snakes and ladders, monopoly is characterized by a discrete time stochastic process with discrete state space.

Introduction: Classification

- **State Space:** Contains all the possible values of X_t and usually denoted by $\mathcal{S}$. It is a discrete space if it is finite or countable otherwise continuous space.

- **Index Set:** Discrete if it has discrete points and continuous if it is an interval of real line.

- **Classification:**

Index set	State Space: Discrete	State Space: Continuous
Discrete	Discrete stochastic process/chain with a discrete state space	Discrete stochastic process with a continuous state space
Continuous	Continuous stochastic process/ chain with a discrete state space	Continuous stochastic process with a continuous state space

Introduction:Classification

- **Realization:** For a particular outcome, $X_t,\ t \in T$ is called a realization or a sample path of the process. That is assignment to each t a possible value of X_t.

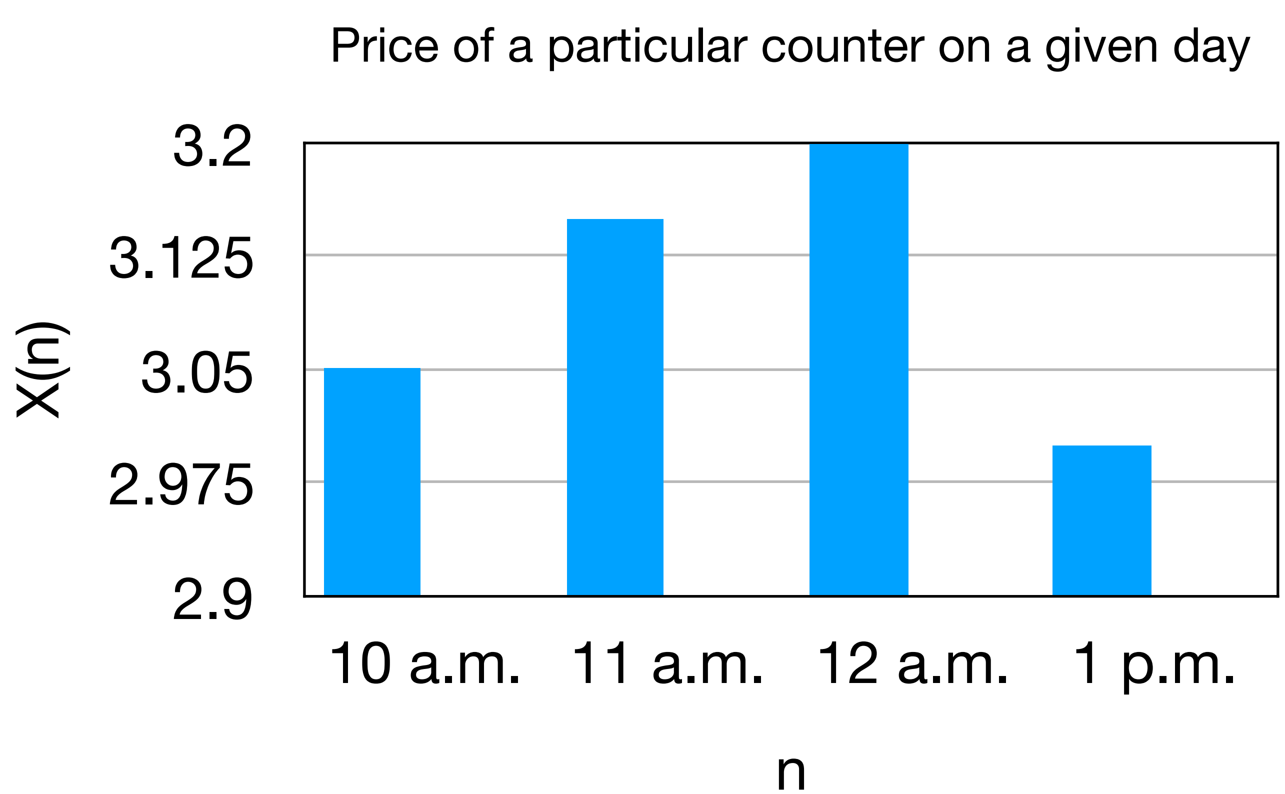
- If the process corresponds to discrete units a realization is a sequence otherwise it is a function of t.

- Examples:

- Successive observation of a tossing a coin. $X_t = 1$ if t^{th} toss is a head and $X_t = 0$ if t^{th} toss is a tail. This is a discrete process with discrete state space.Then some realizations are $1,0,0,1,1,1,\cdots$ and $1,1,1,0,1,0,1,\cdots$.

Introduction:Classification

- The price of a particular stock counter listed on the stock exchange as a function of time. If we are interested in the price at any time t on a given day, then the following is a realization of a continuous process with discrete state space.



Introduction: Exercises

- Find the state space and index set of the following stochastic processes and classify them. For each case describe realizations of the process and graph it.

  - Snakes and ladders board game.

  - The number of telephone calls arriving at an automatic phone-switching system.

  - Number of customers in the time interval $[0,t)$.

  - The n-th bit in the binary expansion of a number in $[0,1)$.

Introduction: Exercises

- **Digital Digital Modulation: Phase-Shift Keying :**

- A basic method for modulation of digital data is phase-shift keying (PSK). In this method, binary data, modeled by a stream of 0's and 1's, is coded onto a carrier frequency by a phase signal.

- Define the random phase $\theta(n)$ as $\theta(n) = \pi/2$ if the n-th bit is 1 and $\theta(n) = -\pi/2$ otherwise. Let T denote the duration of the signal used for each bit.

- Typically, T is a multiple of bit rate (the period of the carrier frequency f_c); that is, $T = m/f_c$ for some integer $m \geq 1$, so that one or more cycles are used per bit. Define the phase signal for the n -th bit as

  $$\Theta(n) = \theta(n), nT \leq t < (n+1)T .$$

  The corresponding transmitted signal is given by

  $$X(t) = \cos(\omega_c t + \Theta(t))$$

  where $\omega_c = 2\pi f_c$ is the carrier frequency in radians/sec.