EM 509: Stochastic Processes

Class Notes

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Characterization of a Stochastic Process

- Partial characterizations for a random variable include its mean, variance, moments, and the like.

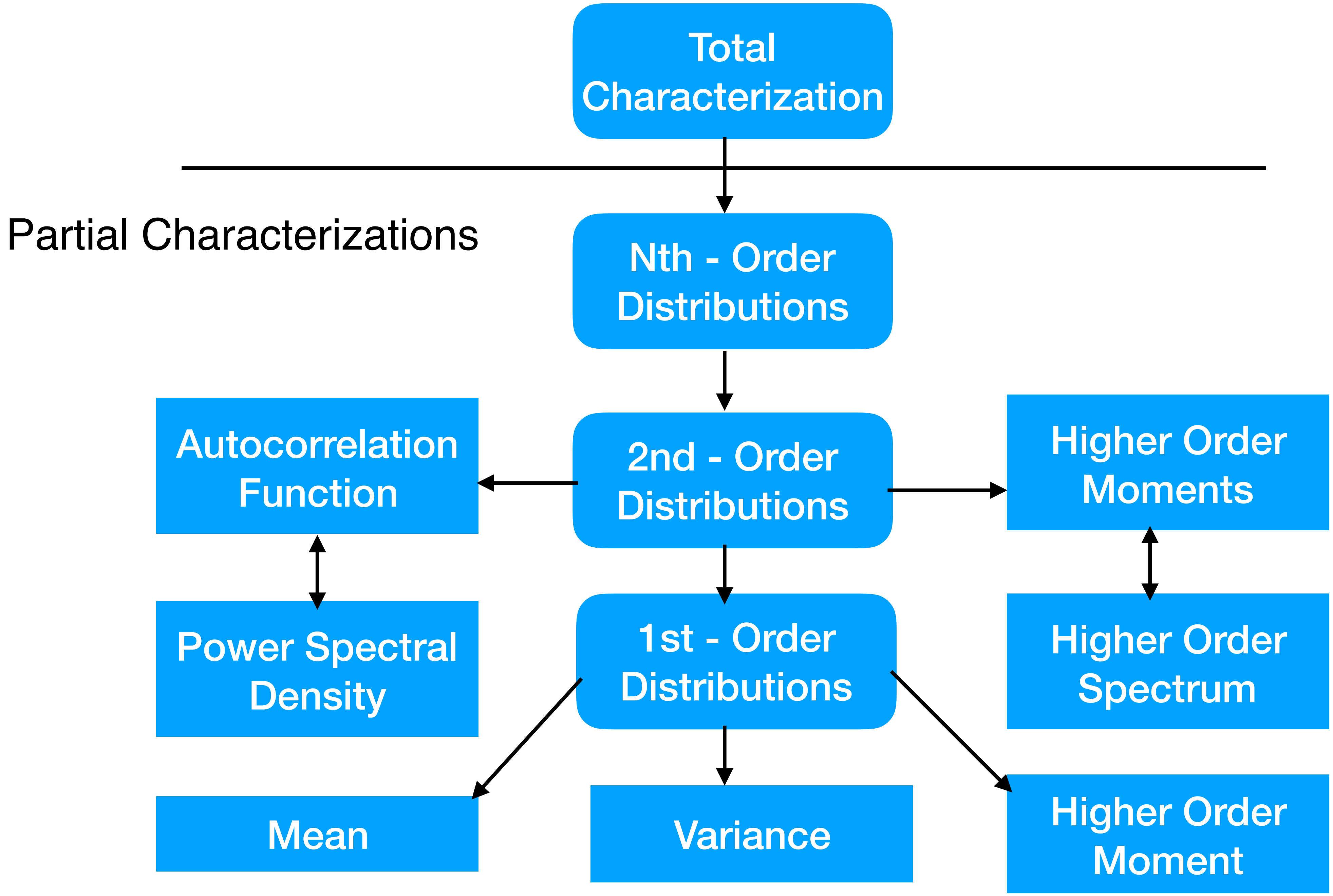
- Similarly, useful partial characterizations for a stochastic process include the mean, variance, and moments. However, all will, in general, be functions of time(index variable)

- For stochastic processes new types of characterizations like the nth-order distributions, autocorrelation function, and spectral density will be defined which will be useful in analyzing random processes. These characterizations give different statistical properties of the process and different amounts of information concerning the process.

- Recall a random variable completely defines the possible events in the underlying probability space. Therefore in the study of stochastic processes all we need is the codomain for the random variable which is called the state space of the process.

Characterization of a Stochastic Process:

A hierarchical relationship between the various types of characterizations



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Characterization of a Stochastic Process: Definitions

Here we will define each of the characterizations and their relationships.

- **Total Characterization of a Stochastic Process:**

- For every t, X_t **or** $X(t)$ is a random variable. This gives us a countably infinite or infinite number of random variables described for the random process.

- A random process is defined to be **completely or totally characterized** if the joint distributions for the random variables $X_{t_1}, X_{t_2}, \cdots, X_{t_n}$ are known for all $t_1, t_2, \cdots, t_n$ and for all n.

Characterization of a Stochastic Process:Definitions

- **First-Order Distributions of a Stochastic Process:**

- The first-order distribution function of a stochastic process is defined as $F_{X_t} = F(x,t) = P(X_t \leq x)$.

- If they are all the same, then $F(X,t)$ does not depend on t and the resulting distribution is called the first-order distribution of the process. Otherwise, there is a family of first order distributions.

- The first-order densities and mass functions are respectively denoted by $f(x,t)$ **and** $p(x,t)$.

Characterization of a Stochastic Process:Definitions

- **Second-Order Distributions of a Stochastic Process:**

- For any t_1, t_2 , let the random variables $X(t_1) = X_1$ **and** $X(t_2) = X_2$. The second-order distribution function for the process $X(t)$ is defined as $F_{X_1 X_2} = F(x_1, x_2 : t_1, t_2) = P(X_1 \leq x_1, X_2 \leq x_2)$.

- The second-order densities and mass functions are respectively denoted by $f(x_1, x_2; t_1, t_2)$ **and** $p(x_1, x_2; t_1, t_2)$.

- The second-order distribution, density and mass functions satisfy the usual properties of joint distribution, density and mass functions.

- In a similar fashion Nth-order distributions can be defined.

Characterization of a Stochastic Process:Definitions

- **Mean and Variance of a Stochastic Process:**

- The mean of a random process is given by $\mu_X(t) = E(X_t)$ .The mean is in general a function of t and it reflects the average behaviour of the process with t.

- The variance of a random process is given by $\sigma_X^2(t) = E(X_t^2) - [E(X_t)]^2$ and is a function of t in general.

- **Higher-order Moments:**

- For $X(t_1) = X_1, X(t_2) = X_2, \cdots, X(t_n) = X_n$ any n random variables from a process, the higher order moments include expectations of the expressions such as the following.

$$E(X_1X_2),\ E(X_1X_2\cdots X_n),\ E(X_1^2X_2^2),\ \cdots$$

Characterization of a Stochastic Process:Definitions

- **Autocorrelation Function of a Stochastic Process:**

- Autocorrelation function of a process represents the interrelationship between the random variables $X(t_1)$ **and** $X(t_2)$ generated from the process $X(t)$.

- Autocorrelation function is defined as $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$. Since $X(t_1)X(t_2) = X(t_2)X(t_1)$ the autocorrelation function is symmetric in t_1 **and** t_2; $R_{XX}(t_1, t_2) = R_{XX}(t_2, t_1)$.

- In certain cases, the autocorrelation function is seen to depend only on the difference $\tau = t_1 - t_2$. For this case, a one-dimensional autocorrelation function is defined as

$$R_{XX}(\tau) = E[X(t+\tau)X(t)].$$

Characterization of a Stochastic Process:Definitions

- **Autocovariance Function of a Stochastic Process:**

- The autocovariance function also measures the interrelationship and is defined as,

$$C_{XX}(t_1, t_2) = E[(X(t_1) - \mu(t_1))(X(t_2) - \mu(t_2))] = R_{XX}(t_1, t_2) - \mu(t_1)\mu(t_2)\,.$$

- If the random variables are uncorrelated then the covariance is zero.

- Thus the variance of the process is, $\sigma^2(t) = C_{XX}(t, t) = R_X(t, t) - \mu^2(t)\,.$

- The normalized autocovariance function is defined as,

$$\rho_{XX}(t_1, t_2) = \frac{C_{XX}(t_1, t_2)}{\sqrt{C_{XX}(t_1, t_1)C_{XX}(t_2, t_2)}} = \frac{C_{XX}(t_1, t_2)}{\sigma(t_1)\sigma(t_2)}\,.$$

This is also called correlation coefficient.

Characterization of a Stochastic Process:Definitions

- **Power Spectral Density of a Stochastic Process:**

- A very important class of stochastic processes are those for which the autocorrelation function is a function of the difference $\tau = t_1 - t_2$.
- For such a situation the autocorrelation characterization can be defined in an equivalent way by taking the Fourier Transform of it and is called Power Spectral density.

- Thus the Power Spectral density of a process is defined as

$$S_{XX}(\omega) = F(R_{XX}(\tau)) = \int R_{XX}(\tau) e^{-i\omega\tau} d\tau$$ **for continous case**

$$P_{XX}(\omega) = \sum_{k=-\infty}^{\infty} R_{XX}(\tau) e^{-ik\omega}$$ **for discrete case**

- This formulation roughly describes the region in the frequency domain where power of the process exists and the relative proportion of power at each frequency.

Characterization of a Stochastic Process:Definitions

- **Power Spectral Density of a Stochastic Process:**

- The power spectral density of a given real stochastic process has the following properties:

1. Even function, $S_{XX}(-\omega) = S_{XX}(\omega)$.

2. Real and non-negative, $S_{XX}(\omega) \geq 0.$

3. Fourier transform pairs, $S_{XX}(\omega) = F(R_{XX}(\tau));\ R_{XX}(\tau) = F^{-1}(S_{XX}(\omega))$.

Characterization of a Stochastic Process:Examples

1. In a communication system, the carrier signal at the receiver is modeled by $X_t = \cos(2\pi f t + \Theta)$. Find the mean function and the autocorrelation function if $\Theta \sim$ **unifom**$[-\pi, \pi]$.

The mean function, $E(X_t) = E(\cos(2\pi f t + \Theta)) = \int_{-pi}^{\pi} \frac{\cos(2\pi f t + \Theta)}{2\pi} d\Theta = 0$

The autocorrelation function,

$$R_{XX}(t_1, t_2) = E(X(t_1)X(t_2)) = E[\cos(2\pi f t_1 + \Theta)\cos(2\pi f t_2 + \Theta)].$$

Using the identity, $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$,

$$R_{XX}(t_1, t_2) = \frac{1}{2}[E(\cos(2\pi f(t_1 + t_2) + 2\Theta) + \cos(2\pi f(t_1 - t_2))].$$

The first cosine has expected value zero just as the mean did. The second cosine is nonrandom, therefore

$$R_{XX}(t_1, t_2) = \frac{1}{2}\cos(2\pi f(t_1 - t_2)).$$

Characterization of a Stochastic Process:Examples

2. Find the autocorrelation function of the following process if $Z_i's$ are uncorrelated with zero mean and variance $\sigma^2 \; \forall i$.

$$X_n = Z_1 + Z_2 + \cdots + Z_n, \;\; n = 1,2,\cdots$$

For $m > n$,

$$X_m = Z_1 + Z_2 + \cdots + Z_n + Z_{n+1} + \cdots + Z_m = X_n + Z_{n+1} + \cdots + Z_m .$$

Then,

$$\begin{aligned} R_{XX}(n,m) &= E(X_n X_m) \\ &= E\left[X_n(X_n + \sum_{i=n+1}^{m} Z_i)\right] \\ &= E[X_n^2] + E\left[X_n \sum_{i=n+1}^{m} Z_i\right]. \end{aligned}$$

To analyze the first term on the right, observe that since the $Z_i's$ are zero, mean of $X_n = 0$.
Also, X_n is the sum of uncorrelated random variables.

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Characterization of a Stochastic Process:Examples

Since the variance of the sum of uncorrelated random variables is the sum of the variances we get,

$$E(X_n^2) = \textbf{Var}(X_n) = \sum_{i=1}^{n} \textbf{Var}(Z_i) = n\sigma^2 .$$

Now consider the second term. Since in the double sum $i \neq j$ and since the $Z_i's$ are uncorrelated with zero mean we get

$$E\left[X_n \sum_{i=n+1}^{m} Z_i\right] = E\left[\left(\sum_{j=1}^{n} Z_j\right)\left(\sum_{i=n+1}^{m} Z_i\right)\right] = \sum_{j=1}^{n} \sum_{i=n+1}^{m} E[Z_j Z_i] = 0.$$

Thus

$$E(X_n X_m) = \sigma^2 n \textbf{ for } m > n .$$

Now since $E(X_n X_m) = E(X_m X_n)$, the general result is

$$R_{XX}(n,m) = \sigma^2 \textbf{min}(m,n) \; m,n \geq 1.$$

Characterization of a Stochastic Process:Examples

3. Let X_n denotes an iid Bernoulli process with probability of success and failure $P(X_n = 1) = p$ **and** $P(X_n = 0) = 1 - p$ respectively. Find the mean variance and autocorrelation of the process.

Solution:

Mean: $\mu(n) = E(X_n) = 1.p + 0.(1 - p) = p$.

Variance: $\sigma^2(n) = E(X_n - E(X_n))^2 = (1 - p)^2 p + (0 - p)^2(1 - p) = p(1 - p)$.

Autocorrelation: $R_{XX}(m, n) = E[X_n X_m] = E[X_n]E[X_m] = p^2$.

Exercises

1. Find the mean, variance, autocorrelation and autocovarince of the random process $X(t) = A\cos(2\pi t)$ where A is a random variable.

2. Consider the random process $Y_n = 2X_n - 1$ where X_n is a Bernoulli process with probability of success and failure respectively given by $P(X_n = 1) = p$ **and** $P(X_n = 0) = 1 - p$. Find the possible outcomes for Y_n . By using the example 3 and the properties of mean and variance find the mean, variance and autocorrelation of the process Y_n.

3. With Y_n as in problem 2 and using the example 2, find the mean, variance and the autocorrelation of the one dimensional random walk $Z_n = Y_1 + Y_2 + \cdots + Y_n,\ n = 1,2,\cdots$.

Exercises

4. Let X_t be a random process with mean function $\mu_X(t)$. Suppose that X_t is applied to a linear time-invariant (LTI) system with impulse response h(t) . Show that the mean function of the output process

is given by

$$Y_t = \int_{-\infty}^{\infty} h(t-\theta)X_\theta d\theta$$

$$\mu_Y(t) = \int_{-\infty}^{\infty} h(t-\theta)\mu_X(\theta)d\theta.$$

Hint:

- Interchange expectation and integration by writing the integral as a Riemann sum and use the linearity of expectation. i.e.

$$E(Y_t) = E\left[\int_{-\infty}^{\infty} h(t-\theta)X_\theta d\theta\right] \approx E\left[\sum_i h(t-\theta_i)X_{\theta_i}\Delta\theta_i\right].$$

- Observe that X_θ is a random variable, while for each fixed t **and** θ , the response $h(t-\theta)$ is just a nonrandom constant that can be pulled out of the expectation.