EM 509: Stochastic Processes

Class Notes

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Special Classes of Stochastic Processes

Here we will consider some special classes of random processes for which it is relatively simple to specify the joint probability distribution function for any set of times.

- Independent and Identically Distributed Processes:
  A stochastic process is said to be independent and identically distributed (i.i.d.) if it consists of an infinite sequence of independent and identically distributed random variables. Thus the joint distribution for any sampling times can be expressed as the product of the first order marginal distribution.

$$f_X(x_1, \cdots, x_n; t_1, \cdots, t_n) = \prod_{i=1}^{n} f_X(x_i)$$

$$p_X(x_1, \cdots, x_n; t_1, \cdots, t_n) = \prod_{i=1}^{n} p_X(x_i)$$

where the first order marginals $f_X(x), p_x(x)$ are independent of time.

- The i.i.d. process is perhaps the simplest possible class, since it can be specified completely in terms of a scalar density or mass function.

Special Classes of Stochastic Processes

- Useful properties of an iid process:
  1. Independence:
     Since the r.v.s in an iid process are independent, any two events defined on sets of random variables with non-overlapping indices are independent.

  2. Memorylessness:
     The independence property implies that the iid process is memoryless in the sense that for any time n, the future $X(t_{n+1}), X(t_{n+2}), \cdots$ is independent of the past $X(t_1), X(t_2), \cdots, X(t_n)$.

  3. Fresh start:
     Starting from any time n, the random process $X(t_{n+1}), X(t_{n+2}), \cdots$ behaves identically to the process $X(t_1), X(t_2), \cdots$, i.e., it is also an iid process with the same distribution. This property follows from the fact that the random variables are identically distributed (in addition to being independent).

Special Classes of Stochastic Processes

- Useful properties of an iid process:

  4. An i.i.d random process is strictly stationary.

$$f_X(x_1, \cdots, x_n; t_1, \cdots, t_n) = \prod_{i=1}^{n} f_X(x_i) = f_X(x_1, \cdots, x_n; t_1 + \epsilon, \cdots, t_n + \epsilon)$$

$$p_X(x_1, \cdots, x_n; t_1, \cdots, t_n) = \prod_{i=1}^{n} p_X(x_i) = p_X(x_1, \cdots, x_n; t_1 + \epsilon, \cdots, t_n + \epsilon)$$

Therefore iid processes enjoys all the nice properties of SSS processes.

Special Classes of Stochastic Processes

- Gaussian Stochastic Processes:

  A stochastic process is said to be Gaussian stochastic process if the random variables $X(t_1), \cdots, X(t_n)$ are jointly Gaussian for all choices of $t_1, \cdots, t_n$ and for all n.

  Recall that the probability density function of jointly Gaussian random variables is determined by the vector of means and by the covariance matrix, as

$$f_X(x_1, \cdots, x_n, t_1, \cdots, t_k) = \frac{e^{-1/2(\underline{x}-\underline{\mu}_X)^T \Sigma_X^{-1}(\underline{x}-\underline{\mu}_X)}}{(2\pi)^{n/2}(|det\Sigma_X|)^{1/2}}$$

  where

$$\underline{x} = (x_1 \ \cdots \ x_n)^T, \ \mu_X = (\mu_X(t_1) \ \cdots \mu_x(t_n))^T, \ \Sigma_X = \begin{pmatrix} C_{XX}(t_1, t_2) & \cdots & C_{XX}(t_1, t_n) \\ \vdots & \vdots & \vdots \\ C_{XX}(t_1, t_n) & \cdots & C_{XX}(t_n, t_n) \end{pmatrix}.$$

  Thus, Gaussian random processes are completely specified by the process mean and autocovariance function.

Special Classes of Stochastic Processes

- Useful properties of Gaussian processes:

  Many important practical random processes are subclasses of normal random processes. Furthermore, Gaussian processes have additional properties which make them particularly useful.

  1. The mean and autocorrelation functions completely characterize a Gaussian random process.

  2. An important property of Gaussian processes is that wide sense stationarity and strict sense stationarity are equivalent. That is a random process that is wide sense stationary and Gaussian is strict sense stationary.

  3. Integral and derivative of a Gaussian process is also a Gaussian random processes.

Special Classes of Stochastic Processes

- Process with Independent Increments

A stochastic process $X(t)$ is called an independent increments process if for all $s < t$ the random variables $X(t) - X(s)$ **and** $X(\tau)$ are independent for any $\tau \leq t$.

- In particular, this implies that if $t_1 \leq t_2 \leq \cdots$ then the increments of the process $X(t_1), X(t_2) - X(t_1), X(t_3) - X(t_2), \cdots$ are independent. This property makes it easier to compute the joint probability mass and density functions as follows:

$$
\begin{aligned}
p_X(x_1, \cdots, x_n; t_1, \cdots, t_n) &= Pr[X(t_1) = x_1, \cdots, X(t_n) = x_n] \\
&= Pr[X(t_1) = x_1, X(t_2) - X(t_1) = x_2 - x_1 \cdots, X(t_n) - X(t_{n-1} = x_n - x_{n-1}] \\
&= Pr[X(t_1) = x_1] Pr[X(t_2) - X(t_1) = x_2 - x_1] \cdots Pr[X(t_n) - X(t_{n-1}] = x_n - x_{n-1}]
\end{aligned}
$$

due to the independent increments property. Similarly

$$f_X(x_1, \cdots, x_n; t_1, \cdots, t_n) = f_X(x_1; t_1) f_X(x_2 - x_1; t_2 - t_1) \cdots f_X(x_n - x_{n-1}; t_n - t_{n-1}) .$$

Special Classes of Stochastic Processes

- Process with Stationary Increments

A stochastic process $X(t)$ is called a stationary increments process if for $s \le t$, the distribution of $X(t) - X(s)$ and $X(t-s)$ are the same. That is the distribution of $X(t) - X(s)$ does not depend on time.

Note:

It is much harder to characterize processes in continuous time with stationary, independent increments. Random processes indexed by an uncountable set are much more complicated in a technical sense than random processes indexed by a countable set. In spite of the technical difficulties, however, many of the underlying ideas are the same.

Special Classes of Stochastic Processes

- Markov Processes

A process with the property that the probability of any particular future behavior of the process, when its present state is known exactly, is not changed by additional knowledge concerning its past behavior is called a Markov process.

That is for any $t_1 < \cdots < t_n$,

$$Pr(X(t_n) = x_n \,|\, X(t_1) = x_1, \cdots, X(t_{n-1}) = x_{n-1}) = Pr(X(t_n) = x_n \,|\, X(t_{n-1}) = x_{n-1}) \,.$$

- The value $X(t_n)$ is usually called the 'state' and the change from one state to another is called a state transition.
- A Markov process with finite or countable state space is called a Markov chain.
- A Markov process for which all realizations are continuous functions is called a diffusion process.

Special Classes of Stochastic Processes

- iid processes are Markov.

- Processes with independent increments are Markov.

For $t_1 < \cdots < t_{n+1}$, the vector $(X(t_1), \cdots, X(t_n))$ is a function of the increments $X(t_1), X(t_2) - X(t_1), \cdots, X(t_n) - X(t_{n-1})$ and is thus independent of the increment $X(t_{n+1}) - X(t_n)$. But $X(t_{n+1})$ is determined by $X(t_{n+1}) - X(t_n)$ and $X(t_n)$ only. Thus Markov.

Special Classes of Stochastic Processes

- Counting Processes

A process $N(t)$ is called a counting process if $N(t)$ is the (finite) number of "events" occurring in (0, t] satisfying

1. For any t, $N(t) \in \mathbb{N}$.

2. For $s < t, N(s) \le N(t)$.

3. The process $N(t)$ is right continuous.

A counting process is called simple if it never jumps by more than 1. Thus, if $N(t)$ is a simple counting process, then $N(t) - N(s)$ is equal to the number of jumps that occur in (s, t]. We usually refer to the location of a jump as an event.

Special Classes of Stochastic Processes

Some special types of counting processes.

- Poisson process

Suppose that $N(t)$ is a simple counting process with independent stationary increments. If $Pr(N(0)=0)=1$ and $Pr[\forall t N(0)=0)]=0$ then $\forall t, N(t)$ has a poisson distribution with a parameter $\lambda \in (0,\infty)$. Such a process is called a Poisson process.

- Renewal process

This is a generalization of the Poisson process. Let $X(t)$ be an iid process with $X(t) \geq 0, Pr(X=0) < 1$. Let $S_n = \sum_{i=1}^{n} X_i$. Define

$$N(t) = \mathbf{max}\{n; S_n \leq t\};\ t \geq 0.$$

This is called renewal process and S_n is called the nth renewal.

Special Classes of Stochastic Processes: Exercises

1. For each of the following, determine whether the process is WSS and/or SSS.

(a) An iid Bernoulli process with probability of success p.

(b) A process $X_n = A\sin(n\omega_0 + \phi)$ where A is a Gaussian random variable with mean μ_A and variance σ_A^2.

(c) A process $X(n) = A\cos(n\omega_0) + B\sin(n\omega_0)$ where A and B are uncorrelated zero mean random variables with variance σ^2 and ω_0 is a constant.

(d) A process $Y(n) = X(n) - X(n-1)$ where $X(n)$ is an iid Bernoulli process with probability of success p.

2. Consider a random process given by $X(t) = U\cos(\omega_0 t) + V\sin(\omega_0 t)$ where ω_0 is a constant and U, V are random variables.

i) Show that the condition $E(U) = E(V) = 0$ is necessary for $X(t)$ to be SSS.

ii) Show that $X(t)$ is WSS if and only if U and V are uncorrelated with equal variance. That is

$$E(UV) = 0,\ E(U^2) = E(V^2) = \sigma^2.$$