# EM 509: Stochastic Processes

Class Notes

Transform Methods in Stochastic Theory

(Lecture 5)

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# Transform Methods in Stochastic Theory

- We saw in a previous lecture, to provide different statistical properties of the process and different amounts of information concerning the process, we need to calculate various types of characterizations.

- Some types of such characterizations are mean, variance, moments, the nth-order distributions, autocorrelation function, and spectral density etc.

- In stochastic theory we are dealing with an infinite family of random variables.

- Therefore to perform such operations in standard ways will sometimes be possible but, in many situations it will be very tedious or sometimes impossible.

# Transform Methods in Stochastic Theory

For example:

- Suppose we have independent random variables $X_1, X_2, X_n, \cdots$ each has a Poisson distribution with parameter $\lambda_i, i = 1,2,\cdots$.

- Suppose we need to find the distribution of the sum $X_1 + \cdots + X_n$.
  In this case we can use mathematical induction to show that the sum has Poisson distribution with parameter $\lambda_1 + \cdots + \lambda_n$.

- In the above we found a 'natural' way to manipulate the algebra so that we could recognize the answer.

- What would happen if we considered other sums of random variables? Will it be possible to come up with a mathematical tool as above?

- It is nice if we have a procedure that will work in general.

# Transform Methods in Stochastic Theory

- Such an approach exists and it is called the theory of **generating functions or transform methods**.

- Some important transform methods are:

    1. Moment Generating Function

    2. Laplace Transform

    3. Characteristic Function

- Depending on your field of study, you can use the type that will be more effective than the others.

- Here we will study about these generating functions and their relationships.

# Transform Methods in Stochastic Theory

## Moment generating function(MGF)

- For $t \in \mathbb{R}$, the moment generating function of a random variable $X$ is defined as

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum_{k=-\infty}^{\infty} e^{tx_k} P(X_k) & \textbf{discrete random variable} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)dx & \textbf{continuous random variable} \end{cases}$$

where where $f_X, P$ are the probability density or mass function respectively.

- The above is an infinite series or integral. Therefore we need to see whether it exists(finite).

- For example it is clear from the definition that for the integral to exist, the right tail of the density has to go to zero faster than $e^{-x}$.

- This is not the case for fat-tailed distributions.

# Transform Methods in Stochastic Theory

- **Laplace Transform**

- Recall the Laplace transform of a function is defined as

$$F(t) = \mathscr{L}(f(x)) = \int_0^\infty e^{-tx} f(x)dx\,.$$

- Thus if we flip the sign on $t$ in the definition of $M_X(t)$ ,we have the **two-sided Laplace transform** of $f_X$ .

- That is, the moment generating function of $X$ at $t$ is the two-sided Laplace transform of $f_X$ at $-t$ .

- Thus if the density function is zero for negative values, then the two-sided Laplace transform reduces to the more common (one-sided) Laplace transform.

# Transform Methods in Stochastic Theory

- **Characteristic Function**

- The characteristic function of a random variable is a variation on the moment generating function.

- Rather than using the expected value of $tX$ , it uses the expected value of $itX$ .

- This means the characteristic function of a random variable is the Fourier transform of its density/mass function.

- Characteristic functions are easier to work with than moment generating functions.

- Existence is not a problem for the characteristic function because the Fourier transform exists for any density/mass function.

# Transform Methods in Stochastic Theory

- **Characteristic Function**

- For $X$ a random variable and $t \in \mathbb{R}$ the characteristic function is defined as

$$\psi_X(t) = E(e^{itX}) = \begin{cases} \sum_{k=-\infty}^{\infty} e^{itx_k} P(X_k) & \textbf{discrete random variable} \\ \int_{-\infty}^{\infty} e^{itx} f_X(x)dx & \textbf{continuous random variable} \end{cases}.$$

- Thus the characteristic function is the most general form of the transforms.

- Therefore we will study only about the characteristic function and all the results are appropriately applicable for other transforms.

# Transform Methods in Stochastic Theory

- **Characteristic Function**

- Using Euler formula, $E(e^{itx}) = E(\cos tx) + iE(\sin tx)$.

- The above gives the expectation of the complex random variable $e^{itX}$ in terms of expectations of two real random variables.

- Since $|e^{itX}| = 1, E(|e^{itx}|) = E(|e^{itx}|^2) = 1.$

- This is a transformation that transforms probability density function or probability mass function to a complex function.

# Transform Methods in Stochastic Theory

- **Characteristic Function**

- Uniqueness: If two random variables $X_1$ **and** $X_2$ have the same characteristic functions, then they have the same distribution functions.

  i.e if $\psi_{X_1}(t) = \psi_{X_2}(t) \; \forall t \in \mathbb{R}$, then $F_{X_1} = F_{X_2} \; \forall x \in \mathbb{R}$. This is written as

$$X_1 \stackrel{d}{=} X_2 \,.$$

- There are several additional properties that follow immediately from the definition of the characteristic function.

# Transform Methods in Stochastic Theory

## Characteristic Function

- Properties:

1. Characteristic function $\psi_X(t)$ exists for any random variable.

2. At $t = 0, \psi_X(0) = 1$ and $|\psi_X(t)| \leq 1$.

3. Characteristic function $\psi_X(t)$ is uniformly continuous.

4. Characteristic function of $a + bX$ for $a, b$ constants is

   $\psi_{a+bX} = e^{iat}\psi_X(bt)$.

5. Characteristic function of $-X$ is the complex conjugate $\bar{\psi}_X(t)$.

# Transform Methods in Stochastic Theory

- **Characteristic Function**

- Properties:

6. Characteristic function $\psi_X(t)$ is real valued iff $X \overset{d}{=} -X$.
   i.e the distribution is symmetric about zero.

7. For any complex numbers, $z_l;\ l = 1,2,\cdots,n$ and for any real $t_l;\ l = 1,2,\cdots,n$ we have

$$\sum_{l=1}^{n}\sum_{k=1}^{n} z_l \bar{z}_k \psi_X(t_l - t_k) \geq 0.$$

   i.e. the characteristic function is positive semidefinite.

# Transform Methods in Stochastic Theory

- **Characteristic Function**

Examples:

1. Standard Normal Distribution, $X \in N(0,1)$.

$$\psi_X(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi}} e^{x^2/2} dx .$$

Differentiating w.r.t the parameter t and allowing to move differentiation inside the integral sign, we get

$$\psi_X(t)' = \frac{d\psi_X(t)}{dt} = \int_{-\infty}^{\infty} \frac{d}{dt} e^{itx} \frac{1}{\sqrt{2\pi}} e^{x^2/2} dx = \int_{-\infty}^{\infty} ixe^{itx} \frac{1}{\sqrt{2\pi}} e^{x^2/2} dx = \int_{-\infty}^{\infty} - ie^{itx} \frac{1}{\sqrt{2\pi}} (-xe^{x^2/2}) dx .$$

By integration by parts we get

$$\psi_X(t)' = - ie^{itx} \frac{1}{\sqrt{2\pi}} e^{x^2/2} |_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left( -ti^2 e^{itx} \frac{1}{\sqrt{2\pi}} e^{x^2/2} \right) dx = 0 - t\psi_X(t) .$$

# Transform Methods in Stochastic Theory

- **Characteristic Function**

This results in the first order linear ordinary differential equation

$$\psi_X'(t) + t\psi_X(t) = 0.$$

Using the integrating factor we get $\psi_X(t) = Ce^{-t^2/2}$.

Since $\psi_X(0) = 1$ we have $\psi_X(t) = e^{-t^2/2}$.

Thus we have obtained $X \in N(0,1) \Leftrightarrow \psi_X(t) = e^{-t^2/2}$.

Note: Since this is real valued, therefore by the properties of the characteristic function we get $-X \in N(0,1)$.

- **Characteristic Function**

2. Poisson Distribution, $X \in Po(\lambda),\ \lambda > 0.$

$$\psi_X(t) = \sum_{k=o}^{\infty} e^{itk} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=o}^{\infty} \frac{(e^{it}\lambda)^k}{k!} = e^{-\lambda} e^{e^{it}\lambda} = e^{(e^{it}-1)\lambda}.$$

Thus we have obtained

$$X \in Po(\lambda) \Leftrightarrow \psi_X(t) = e^{(e^{it}-1)\lambda}.$$

- **Characteristic Functions and Moments of Random Variables**

- If the random variable $X$ has $E(|X|^K) < \infty$, then

$$\frac{d^k}{dt^k}\psi_X(t)\,|_{t=0} = \frac{d^k}{dt^k}\psi_X(0) = i^k E(X^K)\,.$$

This can be proved changing the order of differentiation and expectation,

$$\frac{d}{dt}\psi_X(t) = E[\frac{d}{dt}e^{itX}] = E[iXe^{itX}]$$
$$\frac{d}{dt}\psi_X(0) = iE[iXe^{itX}]\,.$$

- **Characteristic Functions and Moments of Random Variables**

Example: Mean and Variance of the Poisson Distribution

$$\psi_X(t) = e^{(e^{it}-1)\lambda}$$

$$\frac{d\psi_X(t)}{dt} = e^{(e^{it}-1)\lambda} i\lambda e^{it}$$

$$\frac{d^2\psi_X(t)}{dt^2} = e^{(e^{it}-1)\lambda} i^2\lambda^2 e^{2it} + e^{(e^{it}-1)\lambda} i^2\lambda e^{it}.$$

Thus

$$E(X) = \frac{1}{i}\frac{d}{dt}\psi_X(0) = \lambda$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{1}{i^2}\frac{d^2}{dt^2}\psi_X(0) - \lambda^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

# Transform Methods in Stochastic Theory

- **Characteristic Functions of Sums of Independent Random Variables**

- Suppose $X_1, \cdots, X_n$ are independent random variables with respective characteristic functions $\psi_{X_k}(t), k = 1,2,\cdots, n$.

- Then the characteristic function of their sum $S_n = \sum_{k=1}^{n} X_k$ is given by $\psi_{S_n}(t) = \psi_{X_1}(t)\cdots\psi_{X_n}(t)$.

- Thus if $X_1, \cdots, X_n$ iid random variables with characteristic function $\psi_X(t)$ then

$$\psi_{S_n}(t) = (\psi_X(t))^n .$$

# Transform Methods in Stochastic Theory

- **Central Limit Theorem**

- Suppose $X_1, X_2, \cdots$ is an infinite sequence of iid random variables with

$$E(X_k) = \mu,\ Var(X_k) = \sigma^2;\ k = 1, \cdots .$$

- Standardize each random variable by subtracting the common mean and then dividing the difference by the common standard deviation,

$$Y_k = \frac{X_k - \mu}{\sigma}.$$

- Then $Y_k's$ are iid with $E(Y_k) = 0,\ Var(Y_k) = 1.$

- Now define $W_n$ by adding the first n of the $Y_k's$ and scale the sum by the factor $\frac{1}{\sqrt{n}}$ so that,

$$W_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k = \sum_{k=1}^{n} \frac{Y_k}{\sqrt{n}}.$$

# Transform Methods in Stochastic Theory

- **Central Limit Theorem**

- Note: A useful result:
  If $E(|X|^n) < \infty,$ for some n then

$$\psi_X(t) = 1 + \sum_{k=1}^{n} E(X^k)\frac{(it)^k}{k!} + o(|t|^n).$$

- Next we will compute the characteristic function of $W_n$ .

$$\psi_{W_n}(t) = (\psi_{Y/\sqrt{n}}(t))^n, \ \because Y_k's \ \ iid$$

$$= (\psi_Y(t/\sqrt{n}))^n, \ \because \psi_{Y/\sqrt{n}}(t) = \psi_Y(t/\sqrt{n})$$

- Now expanding $\psi_Y(t/\sqrt{n})$ as given in the above note along with $E(Y_k) = 0, Var(Y_k) = 1$ we get,

$$\psi_Y(t/\sqrt{n}) = 1 - \frac{t^2}{2n} + o(t^2/n).$$

## Central Limit Theorem

- Thus the characteristic function of $W_n$ is given by

$$\psi_{W_n}(t) = (1 - \frac{t^2}{2n} + o(t^2/n))^n .$$

- Now we will see what happens when $n \to \infty$. Taking the limit,

$$\lim_{n\to\infty} \psi_{W_n}(t) = e^{-t^2/2} .$$

- Thus we observe that the characteristic function of $W_n$ converges for all t to the characteristic function of $N(0,1)$. Thus $W_n \in N(0,1)$.

- This is central limit theorem.

# Exercises

1. Prove the property, if the characteristic function of $X$ is $\psi_X(t)$ then the characteristic function of $a + bX,\ a, b$ constants is $\psi_{a+bX} = e^{iat}\psi_X(bt)$.

2. In the notes we obtained the characteristic function of a random variable $Z \in N(0,1)$. Using this and the property given in question 1 above, show that the characteristic function of $X \in N(\mu, \sigma^2)$ is $\psi_X(t) = e^{i\mu t - \sigma^2 t^2/2}$.

3. If $X$ has Bernoulli distribution with probability of success $p$ then show that the characteristic function of $X$ is given by $\psi_X(t) = (1-p) + e^{it}p$.

# Exercises

4. Suppose $X_1, X_2, \cdots, X_n$ are independent random variables and each $X_k \in N(\mu_k, \sigma_k^2);\ k = 1,2,\cdots,n$. Then show that for any constants $a_1, a_2, \cdots, a_n$ by using characteristic functions

$$S_n = \sum_{k=1}^{n} a_k X_k \in N\left( \sum_{k=1}^{n} a_k \mu_k, \sum_{k=1}^{n} a_k^2 \sigma_k^2 \right).$$

Deduce that if $X_1, X_2, \cdots, X_n$ are iid and $N(\mu, \sigma^2)$ then

$$\bar{X} = \frac{1}{n} \sum_{k=1}^{n} X_k \in N\left( \mu, \frac{\sigma^2}{n} \right).$$