EM 509: Stochastic Processes

Class Notes

Stochastic Models

(Lecture 6)

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Stochastic Models

- Now we are done with the basic theory needed for studying stochastic models.

- There are many stochastic models that are important in many applications of practical interest.

- Some examples are Bernoulli processes, Markov processes, Poisson processes, Wiener processes etc.

- In this course we will study on Markov Processes.

- Your project is based on the processes based on Markov processes which has applications in many areas.

Stochastic Models: Markov Processes

- Markov processes or Markov chains are characterized by the dynamical property that they never look back.

- That is as long as we know the present value of a Markov process, the future behavior of the process does not change if additional information about past recordings of the process is provided.

- Based on the nature of the index set, Markov processes are of two types, Discrete time Markov chains and Continuous time Markov chains.

- Some examples of discrete time Markov chains are random walks and branching processes.

- Some examples of continuous time Markov chains are birth–death processes such as queueing theory.

Stochastic Models: Markov Processes

- In general, the index set does not have to describe time but is also commonly used to describe spatial location.

- The state space can be finite, countably infinite, or uncountable, depending on the application.

- Now recall that in order to be able to analyze a stochastic process, we need to make assumptions on the dependence between the random variables.

- Markov processes are based on the most common dependence structure, called Markov property. This property says “conditioned on the present, the future is independent of the past.”

- First we will do discrete time Markov chains and then continuous time Markov chains.

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- Markov property:

  Let $X_1, X_2, \cdots$ be a sequence of discrete random variables, taking values in some set $\mathcal{S}$. The sequence satisfies the Markov property if the probability of any given state X_{n+1} only depends on its immediate previous state X_n. Formally

$$P(X_{n+1} = j \,|\, X_1 = i_1, X_2 = i_2, \cdots, X_{n-1} = i_{n-1}, X_n = i) = P(X_{n+1} = j \,|\, X_n = i)$$

$$\forall i, j, i_1, \cdots, i_{n-1} \in \mathcal{S} \text{ and } \forall n.$$

- Such a sequence X_n is called a discrete time Markov chain.

Stochastic Models: Markov Processes

Discrete time Markov chains

- In general, the probability $P(X_{n+1} = j \mid X_n = i)$ depends on i, j, n.

- But is often the case that there is no dependence on n. We call such chains **time-homogeneous** and we will restrict our attention to these chains only.

- Since the conditional probability in the definition thus depends only on i, j we use the notation,

$$p_{ij} = P(X_{n+1} = j \mid X_n = i),\ i, j \in \mathcal{S}.$$

- These probabilities are called the **transition probabilities** of the Markov chain.

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- Thus, if the chain is in state i , the probabilities p_{ij} describe how the chain chooses which state to jump to next.

- Obviously the transition probabilities have to satisfy the following two criteria:

  1. For all $i, j \in \mathcal{S}, p_{ij} > 0$

  2. For all $i \in \mathcal{S}, \sum_{j \in \mathcal{S}} p_{ij} = 1$

- Next we will consider some examples.

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- Examples

1.Gambler Ruin Problem:

Suppose a gambler has Rs.100. He bets Rs.1 each game, and wins with probability say p. He stops playing once he becomes broke.

- Here the state space is $\mathcal{S} = \{0,1,2,\cdots\}$, and is countably infinite.

- If the chain is in state $i \geq 1$, it can jump to either $i-1$ or $i+1$ according to the transition probabilities

$$p_{i,i+1} = p, \; p_{i,i-1} = 1 - p = q \,.$$

Stochastic Models

Discrete time Markov chains

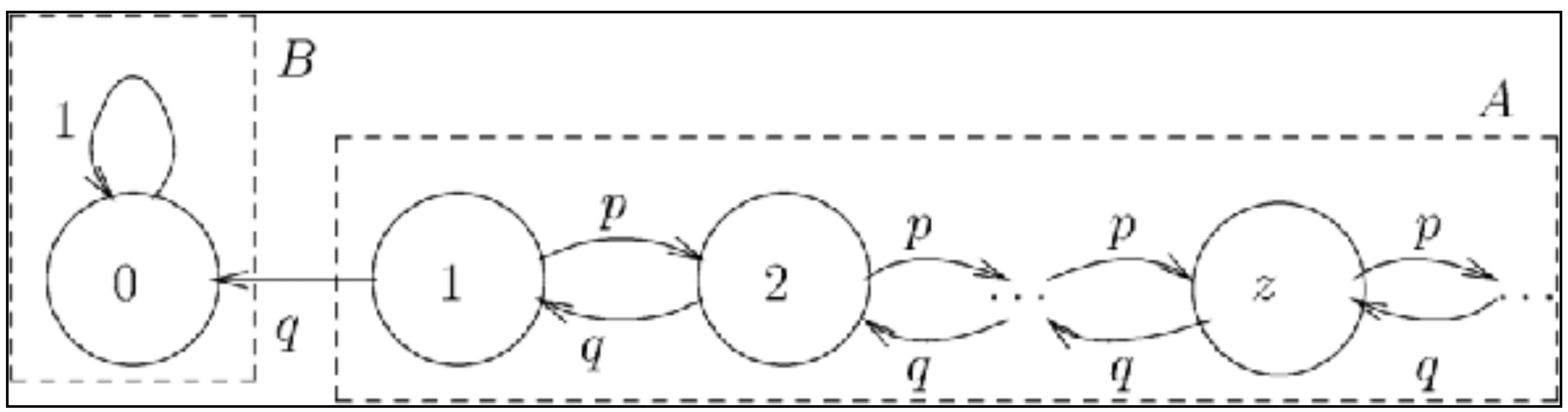
- When $i = 0$, this means that you are ruined and cannot play anymore. Thus, you can jump to 0 but not from it.
- It is customary to describe this by letting $p_{00} = 1$, thus imagining that the chain performs an eternal sequence of jumps from 0 to itself.
- We need a method visualize the dynamics of a Markov chain whenever possible.
- This will be done using transition graphs with nodes (or vertices) representing the states of the Markov chains and edges representing transitions.
- Usually ovals/circles are used to represent the states and arrows show the possible transitions and their corresponding probabilities.

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

  - This graphical way of describing a Markov chain is called a **transition diagram or process diagram.**

  - For this example the transition diagram is given below.



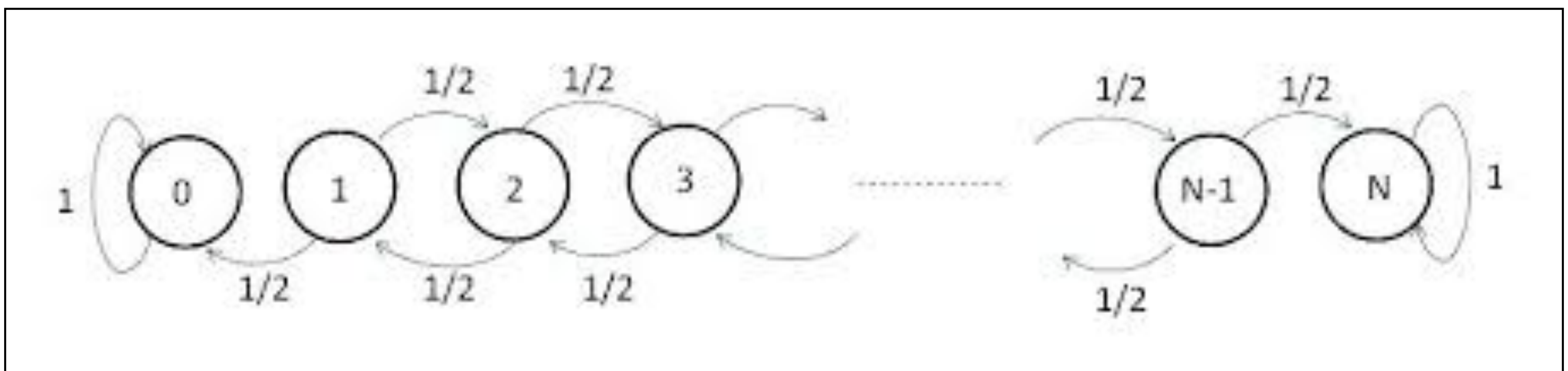
  - Note that the sum of the numbers on the arrows going out from each state is 1. This is the criterion 2 stated earlier.

Stochastic Models: Markov Processes

Discrete time Markov chains

- If we modify the problem by adding the condition that the gambler wins if he earns say Rs N. Then this will be a finite state Markov chain with state space $\mathcal{S} = \{0, 1, \cdots, N\}$.

- Then with $p = q = 1/2$ the transition diagram is as follows.



Stochastic Models: Markov Processes

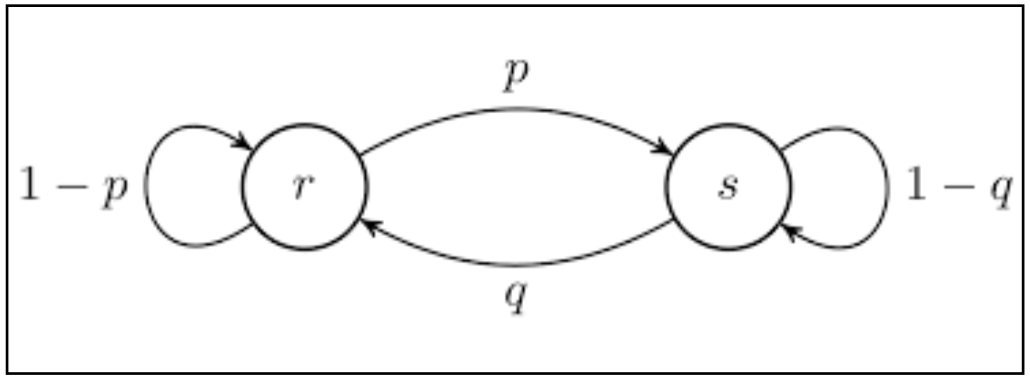
- **Discrete time Markov chains**

2. ON/OFF System:

Consider a system that alternates between the two states OFF(r or 0) and ON(s or 1) and that is checked at discrete time points. If the system is OFF at one time point, the probability that it has switched to ON at the next time point is p, and if it is ON, the probability that it switches to OFF is q.

- State space $\mathcal{S} = \{0,1\} = \{r, s\}$.

- The transition diagram is



Stochastic Models: Markov Processes

- **Discrete time Markov chains**

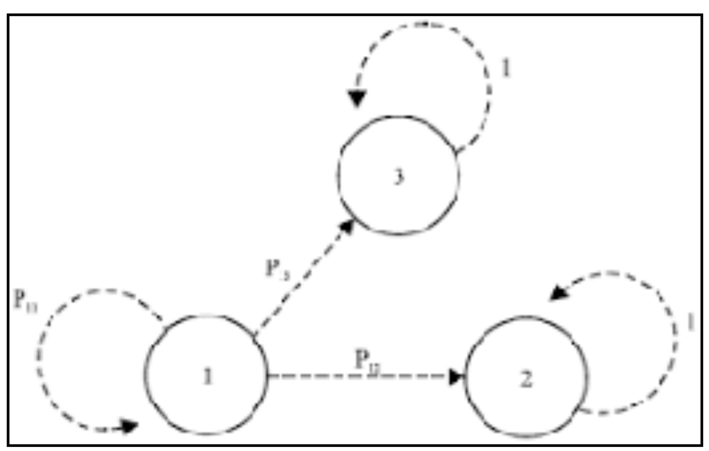
- Examples

3. Problem in Genetics:

A certain gene in a plant has two alleles, A and a. Thus, its genotype with respect to this gene can be AA , Aa , or aa . Now suppose that a plant is crossed with itself and one offspring selected that is crossed with itself and so on and so forth.

- The state space is $\mathscr{S} = \{AA, Aa, aa\}$. The Markov property is clear, since the offspring's genotype depends only on the parent plant, not the grandparent.

- Clearly, genotypes AA and aa can have only themselves as offspring and for the type Aa, using Punnett square $p_{22} = p_{33} = 1,\ p_{12} = p_{13} = 1/4,\ p_{22} = 1/2.$



$1 = Aa,\ 2 = AA,\ 3 = aa\,.$

Stochastic Models: Markov Processes

Discrete time Markov chains

Transition matrix

It is convenient to summarize the transition probabilities in a matrix.

Define a matrix P which has p_{ij} as its $(i, j)th$ entry. The matrix P is called a **transition matrix**.

Depending on the state space, the transition matrix may be finite or infinite.

For example 1, the transition matrix is

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ q & 0 & p & \cdots \\ 0 & q & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

For example 2, the transition matrix is

$$P = \begin{pmatrix} 1-p & p \\ 1 & 1-q \end{pmatrix}.$$

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Time Dynamics of a Markov Chain**

  - The most fundamental aspect of a Markov is how it develops over time.

  - The transition matrix provides us with a description of the stepwise behavior.

  - But suppose that we want to compute the distribution of the chain two steps ahead.

  - Define the n-step transition probabilities as

$$p_{ij}^{(n)} = P(X_n = j \mid X_0 = i)\,.$$

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Time Dynamics of a Markov Chain**

- Consider n=2, $p_{ij}^{(2)} = P(X_2 = j \,|\, X_0 = i)$.

- The law of total probability gives

$$\begin{aligned} p_{ij}^{(2)} &= \sum_{k \in \mathcal{S}} P(X_2 = j \,|\, X_1 = k, X_0 = i) P(X_1 = k \,|\, X_0 = i) \\ &= \sum_{k \in \mathcal{S}} P(X_2 = j \,|\, X_1 = k) P(X_1 = k \,|\, X_0 = i) \\ &= \sum_{k \in \mathcal{S}} p_{ik} p_{kj} \end{aligned} .$$

- In the second equation above we have used the Markov property.

- The last expression says in order to go from i to j in two steps, we need to visit some intermediate step k and jump from there to j .

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Time Dynamics of a Markov Chain**

- Now by the definition of matrix multiplication we see that $p_{ij}^{(n)}$ is the (i, j)th entry of P^2.

- Thus, in order to get the two-step transition probabilities, we square the transition matrix.

- Repeating the argument above gives n-step transition probabilities by nth power of the transition matrix, P^n.

- Thus if $P^{(n)}$ denotes the n-step transition matrix, $P^{(n)} = P^n$.

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Time Dynamics of a Markov Chain**

  - This gives the relation $P^{(n+m)} = P^{(n)} P^{(m)} \; \forall m, n$ **with** $P^{(0)} = I$.

  - This relation is called Chapman–Kolmogorov equations.

  - Element wise, they become

$$p_{ij}^{(n+m)} = \sum_{k \in \mathcal{S}} p_{ik}^{(n)} p_{kj}^{(m)}, \; \forall m, n \textbf{ and } \forall i, j \in \mathcal{S} .$$

  - This says, to go from i to j in n + m steps, we need to visit some intermediate step k after n steps.

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Time Dynamics of a Markov Chain**

Examples:

1.Refer to the ON/OFF system example

(a) Find the n-step transition matrix.

It is convenient to use diagonalization. The eigenvalues of $P; \lambda_1 = 1,\ \lambda_2 = 1 - p - q$. Thus we get

$$P^n = \frac{1}{p+q}\begin{pmatrix} q & p \\ q & p \end{pmatrix} + \frac{\lambda_2^n}{p+q}\begin{pmatrix} p & -p \\ -q & q \end{pmatrix}.$$

(b) If p =3/4 and q =1/2 and the system starts being OFF, what is the probability that it is ON at time n = 3?

This probability is given by the (0,1) entry of $P^{(n)}$:

$$p_{01}^{(3)} = \frac{3/4}{5/4} + \frac{(-3/4)(-1/4)^3}{5/4} \approx 0.61.$$

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Time Dynamics of a Markov Chain**

2. Refer to the genetics example: Find the n -step transition matrix.

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}; \quad P^n = \begin{pmatrix} 1 & 0 & 0 \\ (1-(1/2)^n)/2 & (1/2)^n & (1-(1/2)^n)/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

- The 0s and 1s remain unchanged; the types AA and aa can have offspring only of their own type.

- The probability to find the type Aa declines rapidly with n, indicating that eventually this genotype will disappear.

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Time Dynamics of a Markov Chain**

Note: One interesting aspect of a Markov chain is its long term behavior and for example we will consider the above two examples.

- Genetic example:

$$\lim_{n\to\infty} P^{(n)} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, if we start in state AA or aa, we stay there, and if we start in state Aa, we eventually end up in either AA or aa with equal probabilities.

- ON/OFF example:

$$\lim_{n\to\infty} P^{(n)} = \frac{1}{p+q}\begin{pmatrix} q & p \\ q & p \end{pmatrix}$$

The rows of this matrix are identical and thus, at a later time point, the probabilities that the system is OFF and ON are approximately q/(p + q) and p/(p + q) respectively, regardless of the initial state.

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Time Dynamics of a Markov Chain**

- In the ON/OFF example, the asymptotic probabilities do not depend on how the chain was started.

- We call the distribution $(q/(p+q), p/(p+q))$ on the state space {0, 1} a limit distribution .

- Compare this with the genetics example where no limit distribution exists, since the asymptotic probabilities depend on the initial state.

- A question of general interest is when a Markov chain has a limit distribution.

- To be able to answer this, we need to introduce some criteria that enables us to classify Markov chains.

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Classification of States**

1. If $p_{ij}^{(n)} > 0$ for some n , we say that state j is **accessible** from state i , written i → j . If i → j and j → i , we say that i and j **communicate** and write this i ↔ j .

   - If j is accessible from i, means that it is possible to reach j from i but not that this necessarily happens.

   In gambler's ruin example, since $p_{12} > 0, 1 \to 2$, but if the chain starts in 1, it may jump directly to 0 , and thus it will never be able to visit state 2 .

   In this example, all nonzero states communicate with each other and 0 communicates only with itself.

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Classification of States**

  - In general, if we fix a state i in the state space , we can find all states that communicate with i and form the communicating class containing i. In this class not only does i communicate with all states but they all communicate with each other.

  - By convention, every state communicates with itself (it can "reach itself in 0 steps") so every state belongs to a class.

  - More precisely relation "↔" is an equivalence relation and thus divides the state space into equivalence classes that are precisely the communicating classes.

  In the gambler's ruin example, there are two classes $C_0 = \{0\},\ C_1 = \{1,2,\cdots\}$.

  In the ON/OFF example, there is only one class, the entire state space.

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Classification of States**

2. If all states in $\mathcal{S}$ communicate with each other, the Markov chain is said to be **irreducible**.

3. Consider a state $i \in \mathcal{S}$ and let τ_i be the number of steps it takes for the chain to first visit the state. Thus

$$\tau_i = \textbf{min}\{n \geq 1 \,|\, X_n = i\}\ .$$

   If $\tau_i = \infty$, the state is never visited.
   If the probability $P(\tau_i < \infty) = 1$, then the state is said to be **recurrent** and if $P(\tau_i < \infty) < 1$, it is said to be **transient**.

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Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Classification of States**

  - A recurrent state thus has the property that if the chain starts in it, the time until it returns is finite.

  - For a transient state, there is a positive probability that the time until return is infinite, meaning that the state is never revisited.

  - This means that a recurrent state is visited over and over but a transient state is eventually never revisited.

  - In an irreducible Markov chain, either all states are transient or all states are recurrent. Thus we can classify the entire Markov chain as transient or recurrent by checking only one state.

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Classification of States**

  - If state space is finite, state i is transient iff there exists state j s.t. j is accessible from i but not other way. That is $i \to j$, **but** $i \nrightarrow j$.

  - If a Markov chain has finite state space, there is at least one recurrent state.

  - A state that with the property that once the chain is there, it can never leave is called an **absorbing** state.

  - For example in the gambler's ruin problem state 0 is trivially recurrent since if we start there, we are stuck there forever. That is $\tau_0 \equiv 1.$ Also it is an absorbing state.

    For state 1 , if we start there and the first jump is to 0 , we never return to1. Thus $\tau_1 = \infty$ **and** $P(\tau_1 < \infty) < 1.$ Thus state 1 is transient and since it communicates with the states 2, 3, ... , they are all transient.

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Classification of States**

4. The period of state i is defined as the greatest common divisor of lengths of cycles through which it is possible to return to i . That is

$$d(i) = \textbf{gcd}\ \{n \geq 1 : p_{ii}^{(n)} > 0\}\ .$$

If $d(i) = 1$, i is called **aperiodic** otherwise it is called **periodic** .

For example consider the ON/OFF example, with $p = q = 1.$

For state 0, $p_{00}^{(n)} > 0$ whenever n is even and 0 otherwise. Thus the set of n s.t $p_{00}^{(n)} > 0$ is $\{2,4,6,\cdots\}$.

Thus, the period of state 0 is 2, which means that the only possible return paths to state 0 have lengths that are multiples of 2. State 1 is also of period 2.

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Stationary Distributions**

- Let P be the transition matrix of a Markov chain with state space $\mathcal{S}$. A probability distribution $\pi = (\pi_1, \pi_2, \cdots)$ on $\mathcal{S}$ satisfying $\pi P = P$ is called a **stationary distribution** of the chain.

  - The entries of π thus satisfy

$$\pi_j = \sum_{i \in \mathcal{S}} p_{ij} \pi_i \ \ \forall \ \ j \in \mathcal{S}, \ \ \sum_{i \in \mathcal{S}} = 1.$$

  - The intuition behind the probability π_j is that it describes what proportion of time that is spent in state j in the long run.

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Stationary Distributions**

  - Note that a stationary distribution may not always exist and there may be more than one.

  - For an irreducible Markov chain if a stationary distribution exists then it is unique.

  - If the state space is finite and the Markov chain is irreducible, then a unique stationary distribution exists.

  - In the ON/OFF example, let $\pi = (\pi_0 \pi_1)$ .Then $\pi P = \pi$ gives

$$\begin{aligned} (1-p)\pi_0 + q\pi_1 &= \pi_0 \\ p\pi_0 + (1-q)\pi_1 &= \pi_1 \end{aligned}.$$

  Solving these with $\pi_0 + \pi_1 = 1$ we get $\pi = \left(q/(p+q), p/(p+q)\right)$.

Stochastic Models: Markov Processes

- **Discrete time Markov chains**

- **Stationary Distributions**

- I mainly illustrated the concept presented in this lecture notes only for ON/OFF example. For home work do the same for other two examples considered in this lecture note.