Poisson Processes

Stochastic Processes Project

Pankayaraj Pathmanathan

Department of Computer Engineering

University of Peradeniya

Contents

¡

1 Introduction

Poisson process is a distribution that is used model the time for an arrival into a system. It can be considered as a continuous version of a binomial process where as in a binomial process the arrival can only happen in an integer multiple of the increment step but in poisson process the arrival can happen at any time with the probability at a single time point is zero. Poisson process can be interpreted as an arrival process with three alternative definitions

1.1 Arrival Process

An arrival process is a sequence $T_1, T_2, T_3, ...$ which specifies the occurrence time of the each consecutive arrival where $T_i < T_{i+1}$. Equivalently it can also be defined by the inter arrival time between two times as $T_{i+1} - T_i$. Another alternative definition can be defined as a counting process with the random variable being the number of arrivals $N(t)$ that happens in a particular time t. The counting process definition and the initial arrival time definition can be related by

$$\{N(t) > n\} = \{T_n > t\} \tag{1.1}$$

1.2 Definition (Inter Arrival Time)

Lets consider the perspective of inter arrival times where $t_i = T_i - T_{i-1}$. This process $t_1, t_2, ...$ is called a poisson process of the inter arrival times are IID random variables (renewal process) and they are exponentially distributed. The pdf if this continuous distribution is given by

$$P\left(t = i\right) = \lambda e^{-\lambda i} \tag{1.2}$$

λ is called the rate parameter when the poisson process ins considered in a one dimensional line.

The mean of the distribution is givien by $\frac{1}{\lambda}$ and the variance is given by $\frac{1}{\lambda^2}$

1.3 Definition (Arrival Time)

If we consider the perspective of an arrival time $T_1, T_2, ...$ Here T_n is the summation of the independent random variables $t_{1],t_2,...}$. Thus the pdf of T_n can be obtained by convolving the distributions. Thus Erlang density can be obtained as

$$P(T_n = t = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \tag{1.3}$$

See A for further details

1.4 Definition 3 (Counting Process)

If we consider the definition of a counting process with the average arrival rate in a unit time the pmf of this discrete distribution is given by

$$P(N(t) = n) = (\lambda t)^n \frac{e^{-\lambda t}}{n!} \tag{1.4}$$

See B for further details.

1.5 Properties

1.5.1 Memoryless

The random variable t is independent of the past. That is

$$P\left(t > t_i + t_j\right) = P\left(t > t_i\right).P\left(t > t_j\right) \tag{1.5}$$

2 Markov Modelling

2.1 Markov Property

The Markov property of the sequence of arrival time $T_1, T_2, ...$ can be proved by considering the increment definition. By definition the increments $t_1, t_2, ..$ such that $t_i = T_i - T_{i-1}$ are independent if it's a poission distribution.

$$\text{P}\ (T_i - T_{i-1} = k_i - k_{i-1} | T_0 = k_0, T_1 - T_0 = k_1 - k_0 ... T_{i-1} - T_{i-2} = k_{i-1} - k_{i-2})$$
$$= \text{P}\ (T_i - T_{i-1} = k_i - k_{i-1})$$
$$= \text{P}\ (T_i = k_i | T_{i-1} = k_{i-1})\ (2.1)$$

Also given the memory less property using the independent increments property p oisson distribution of arrival times $T_1, T_2, ...$ can be modelled as continuous time markov chain. That is this can be seen as the process spends time in a counting state with an exponential distribution and leaves to the next incremental state.

2.2 Markov Formation

A poisson distribution can be formulated as a markov process by considering a birth process. Where the probability of state transition from n to $n + 1$ is 1.

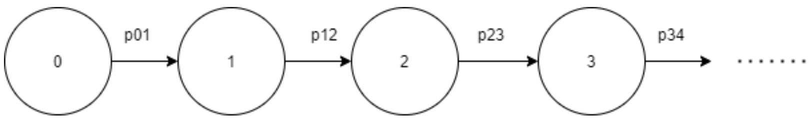

Figure 1: Transition Diagram for Poisson Markov Formation

As in figure

$$p_{i,i+1} = 1 \tag{2.2}$$

$$p_{i,i} = 0 \tag{2.3}$$

$$\lambda_i = \lambda \tag{2.4}$$

2.2.1 A simple Continuous Markov Chain - Drawing Parallels

A time homogenous markov chain is defined as a markov chain that satisfies both the properties of time homogeneity and markov property . That is

$$P\left\{X(t) = j \mid \mathcal{I}_{X(s)}\right\} = P\{X(t) = j \mid X(s)\} \tag{2.5}$$

$$P\{X(t) = j \mid X(s)\} = P\{X(t - s) = j \mid X(0)\} \tag{2.6}$$

Here $I_{X(s)}$ denotes all the information of what came before s. If the process follows these properties then the waiting time T_x for that process in a certain state x would be exponentially distributed due to the memory less property which can be implied by both 2.5, 2.6 as follows

$$\begin{aligned} &P\{T_x > s+t \mid T_x > s\} \\ &= P\{X(r) = x \text{ for } r \in [0, s+t] \mid X(r) = x \text{ for } r \in [0, s]\} \\ &= P\{X(r) = x \text{ for } r \in [s, s+t] \mid X(r) = x \text{ for } r \in [0, s]\} \\ &= P\{X(r) = x \text{ for } r \in [s, s+t] \mid X(s) = x\} \quad \text{(Markov Proptery)} \\ &= P\{X(r) = x \text{ for } r \in [0, t] \mid X(0) = x\} \quad \text{(Time homogeneity)} \end{aligned} \tag{2.7}$$

Thus an alternate definition for Continuous time homogeneous markov process can be derived as a process that satisfies the following

$$\begin{aligned} P\{X(t+h) = x \mid X(t) = x\} &= 1 - \lambda(x)h + o(h) \\ P\{X(t+h) = y \mid X(t) = x\} &= \lambda(x, y)h + o(h) \end{aligned}$$

(2.8)

Where $\lambda(x, y)$ denotes the multiplication of transition probability and the waiting time for a single event following the exponential distribution.

Here we can see both the memory less and homogenity property appears in both poisson and time homogeneous continuous markov process leading to the exponential distribution of wating times. Now if we draw a parallel for this with the poisson distribution it can be shown that the poisson distribution is the simplest continuous time markov process. Now let's consider the discrete transition matrix corresponding to the transition diagram.

$$p = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & . & . \\ 0 & 0 & 1 & 0 & & & & & \\ 0 & 0 & 0 & 1 & & & & & \\ . & & & & . & & & & \\ . & & & & & . & & & \\ . & & & & & & . & & \end{bmatrix} \tag{2.9}$$

And since we are considering poisson processes with stationary rates in this the $\lambda(x, y)$ is a constant λ. Thus the poisson process

$$P\{N(t+h) = j+1 \mid N(t) = j\} = \lambda h + o(h) \tag{2.10}$$

can be defined as a continuous time markov process by the rate parameter λ and the transition matrix 2.9.

2.2.2 Generator

A generator is a matrix formed for a continuous markov chain 'following the Kolmogorov forward equations. The generator matrix can thus me optained as

$$G = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & . & . & . & . & . \\ 0 & -\lambda & \lambda & 0 & & & & & \\ 0 & 0 & -\lambda & \lambda & & & & & \\ . & & & & . & & & & \\ . & & & & & . & & & \\ . & & & & & & . & & \end{bmatrix} \tag{2.11}$$

This completely defines the possion process that's modelled a time homogeneous continuous markov chain.

2.3 Properties

2.3.1 Recurrent/ Transient

The recurrent and transient nature of the continuous Markov chain is same as that of the embedded discrete markov chain. In this case every state in the finite state space can reach the states that are greater in value than them selves but not vice versa. So by definition the all the states in this markov chain are transient.

2.3.2 Irreducible or reducible

No lets look at the states. There are no communicating states. That is you can access a state i for j but you cant access the sate i from j for some j ¡ i. Thus this chain would have countably infinite communication classes namely 1, 2, ... Thus thus chain is a reducible chain.

2.3.3 Stationarity

The stationary distribution of continuous markov chain is given by the stationary distribution of the embedded discrete markov chain. That is also equivalent to $\pi.G = 0$ where G is the generator matrix. For the generator matrix of poisson the only possible solution is $\pi = [0, 0, 0.....]$ which is not a state distribution as the vector don't sum to 1. Therefore for this process **there exists no stationary distribution .**

3 Applications

3.1 Computer Networks

In computer networks the packet arrival times can be modelled as a poisson process. Here the packet arrival times α_k is treated as the poisson arrival times. And the packet length L_k is considered as the inter arrival time. That is it is assumed that the whole transportation medium is used for that packet alone and used to transmit the packet when the packet arrives.

Another type of modelling follwoing [1] in the random access MAC protocols in networking. A packet that suffers collision stays in the network and makes a transmission attempt again until it's successful. These packets are called the backlogs. If the no of arriving packets is denoted by A_k and the backlogs of time k are B_k and the $D_k \in \{0, 1\}$ is the number of departure. Then the backlogs at time $k + 1$ can be written as

$$B_{k+1} = A_k + B_k - D_k \tag{3.1}$$

Here the arrivals A_k at time slot k can be modelled as Poisson process with rate and if the backlogs attempt retransmission with probability p that can be considered as discrete time markov process. And a drift $d(n)$ can be defined as the average change in backlog at a time slot k. Since there can be only one since there can only be one trasnimssion at a time slot. With no new arirval the backlog decreases by 1. If exactly 1 arrival happens and one backlog attempts retrasnmission then the backlog increases by 1. And if m arrivals happens the backlog increases by m. Then we can formalize the equation as

$$\begin{array}{l} \Pr\left(A_k - D_k = +1 \mid B_k = n\right) = \lambda e^{-\lambda}\left(1 - (1-r)^n\right) \\ \Pr\left(A_k - D_k = +m \mid B_k = n\right) = \frac{\lambda^m}{m!} e^{-\lambda} \text{ for } m \geq 2 \\ \Pr\left(A_k - D_k = -1 \mid B_k = n\right) = e^{-\lambda} nr(1-r)^{n-1} \end{array} \tag{3.2}$$

Thus $d(n)$ can be obtained as

$$\begin{aligned} d(n) &= \lambda e^{-\lambda}\left(1 - (1-r)^n\right) + \textstyle\sum_{\infty}^{m=2} m \frac{\lambda^m}{m!} e^{-\lambda} - e^{-\lambda} nr(1-r)^n \\ &= \textstyle\sum_{\infty}^{m=0} m \frac{\lambda^m}{m!} e^{-\lambda} - \lambda e^{-\lambda(1-r)n-1} \\ &= \lambda - e^{-\lambda}(1-r)^n \left(\lambda + \frac{nr}{1-r}\right) \end{aligned} \tag{3.3}$$

Thus modelling as a Poisson process gives us a analytical equation for the change in backlogs.

3.2 Shot Noise

Here following [2] the electrons that arrive at the start of a wire is modelled as a poisson process with a rate λ. And with the help of the modelling it is possible to calculate the current in the wire. That is t seconds after the arrival of electron the current can be obtained as $I(t) = e^{-\beta t}$ for some $/beta > 0$.Now if we have the arrival times of electrons $T_1, T_2....T_n$ then we can model the expected current in the wire as $X(t) = \sum_{i=1}^{N(t)} I\left(t - T_i\right)$. Thus modelling with poisson process can be used to find the current in the wire.

References

[1] A. Kumar, D. Manjunath, and J. Kuri. *Communication Networking: An Analytical Approach.* Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 2004.

[2] S. I. Resnick. *Adventures in Stochastic Processes.* Birkhauser Verlag, CHE, 1992.

A Erlang Distribution

A.1 Derivation

The erlang distribution corresponding to T_{k+1} can be obtained by convolution of the distribution of T_k and the exponential distribution of waiting time t as follows

$$f_{T_{k+1}}(t) = \int_0^t \lambda e^{-\lambda(t-s)} \cdot \lambda^k e^{-\lambda t} \frac{t^{k-1}}{(k-1)!} ds \quad = \lambda^{k+1} e^{-\lambda t} \int_0^t \frac{s^{k-1}}{(k-1)!} ds$$
$$= \lambda^{k+1} e^{-\lambda t} \tfrac{t^k}{k!}$$

(A.1)

Similarly the Erlang distribution of T_2 can be obtained by the convolution of two exponential distributions.

A.2 Mean and Variance

The mean can be obtained as

$$E(T) = E(T_1) + \cdots + E(T_n) = \frac{n}{\alpha} \tag{A.2}$$

Since the arrival times are independent

$$\mathrm{Var}(T) = \mathrm{Var}(T_1) + \cdots + \mathrm{Var}(T_n) = \frac{n}{\alpha^2} \tag{A.3}$$

B Counting Process

B.1 Mean and Variance

For the notational ease in this section only consider as the expected number of arrivals at time interval t. The mean can be derived as

$$\mathrm{E}(X) = \lambda e^{-\lambda} \sum_{t \geq 1} \frac{1}{(t-1)!} \lambda^{t-1}$$

with $j = t - 1$

$$= \lambda e^{-\lambda} \sum_{j \geq 0} \frac{\lambda^j}{j!}$$
$$= \lambda e^{-\lambda} e^{\lambda}$$
$$= \lambda$$

(B.1)

The second moment is given as

$$\begin{aligned}
\mathrm{E}\left(X^2\right) &= \sum_{t\geq 0} t^2 \frac{1}{k!}\lambda^t e^{-\lambda} \\
&= \lambda e^{-\lambda} \sum_{t\geq 1} t \frac{1}{(t-1)!}\lambda^{t-1} \\
&= \lambda e^{-\lambda} \left(\sum_{t\geq 1} (t-1)\frac{1}{(t-1)!}\lambda^{t-1} + \sum_{t\geq 1} \frac{1}{(t-1)!}\lambda^{t-1} \right) \\
&= \lambda e^{-\lambda} \left(\lambda \sum_{t\geq 2} \frac{1}{(t-2)!}\lambda^{t-2} + \sum_{t\geq 1} \frac{1}{(t-1)!}\lambda^{t-1} \right)
\end{aligned}$$

with $i = t-1$ and $j = t-2$

$$\begin{aligned}
&= \lambda e^{-\lambda} \left(\lambda \sum_{i\geq 0} \frac{1}{i!}\lambda^{i} + \sum_{j\geq 0} \frac{1}{j!}\lambda^{j} \right) \\
&= \lambda e^{-\lambda} \left(\lambda e^{\lambda} + e^{\lambda} \right) \\
&= \lambda(\lambda+1) \\
&= \lambda^2 + \lambda
\end{aligned} \tag{B.2}$$

The the variance can be given as

$$\begin{aligned}
\mathrm{var}(X) &= \mathrm{E}\left(X^2\right) - (\mathrm{E}(X))^2 \\
&= \lambda^2 + \lambda - \lambda^2 \\
&= \lambda
\end{aligned} \tag{B.3}$$