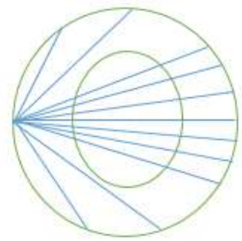
Faculty of Engineering
EM 509 – Stochastic Processes
Answer Sheet
Review of Basic Probability

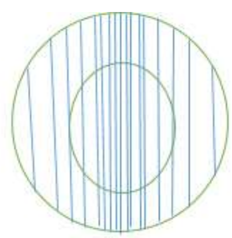
1) **First mode: Random angle**



Take an arbitrary point P on the boundary of the disc. The set of all lines through that points are parameterized by an angle ϕ . The line has to lie within a sector of 60^0 within a range of 180^0.

The probability of this chord intersects the concentric circle of radius 1 $= \frac{60^0}{180^0} = \frac{1}{3}$

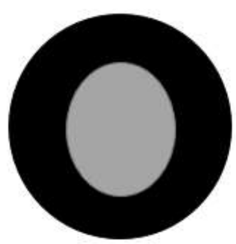
Second mode: Random translation



Take all liens perpendicular to a fixed diameter. If the point of intersection lies on the middle half of the diameter.

The probability of this chord intersects the concentric circle of radius 1 $= = \frac{1}{2}$

Third mode: Random area



If the midpoints of the chords lie in a disc of radius1/2, because the disc has a radius which is half the radius of the unit disc.

probability of this chord intersects the concentric circle of radius 1 $= \frac{\pi 1^2}{\pi 2^2} = \frac{1}{4}$

2) a) Direct measure

$F = P(\Omega)$ $x_0 \in \Omega$ fixed

$$\delta_{x_0} = \begin{cases} 1; x_0 \in A \\ 0; x_0 \notin A \end{cases}$$

I. $\delta_{x_0} \geq 0$ for $A \in F$

II. $\delta_{x_0}(\phi) = 0$

III. A and A^C are disjoint events

$\delta_{x_0}(A \cup A^c) = \delta_{x_0}(\Omega) = 1 = \delta_{x_0}(A) + \delta_{x_0}(A^C)$

Therefore, δ_{x_0} is a measure.

b) Counting measure

$F = P(\Omega), \mu(A) =$ Cardinality of $A = |A|$

I. $\mu(A) \geq 0$ for all $A \in F$

II. $\mu(\phi) = 0$

III. Consider the following 3 cases and let $\{A_1, A_2, ..., A_J, ...\}$ be a countable set of disjoint elements for $j \in \mathrm{N}$

Case I

Assume that all A_j's are finite and only finitely many A_j's are non-empty

$$\sum_{j \in \mathrm{N}} \mu(A_j) = \sum_{j \in \mathrm{N}} |A_j| = \left| \bigcup_{j \in \mathrm{N}} A_j \right| = \mu\left(\bigcup_{j \in \mathrm{N}} A_j \right)$$

Case II

Assume that the cardinality of at least one A_j is infinite such that $\mu(A_k) = \infty$

$$\mu(\bigcup_{j \in \mathrm{N}} A_k) = \infty = \mu(A_k) + \sum_{j \in \mathrm{N}, j \neq k} \mu(A_j) = \sum_{j \in \mathrm{N}} \mu(A_j)$$

Case III

Assume that infinitely many A_j's are non-empty. The cardinality of $\bigcup_{j \in \mathrm{N}} A_j$ is infinite due to A_j's are disjoint.

$$\mu(\bigcup_{j \in \mathrm{N}} A_k) = \infty = \sum_{j \in \mathrm{N}} \mu(A_j)$$

Therefore, $\mu(A)$ is a measure.

n – total number of communicating channels

ρ - Communicationg rate of each channels

3) k - number of used channels

p - the probability of each of the channels fails

R - a random communicating rate

Suppose $i \in \{1,2,...,n\}$ denote a given channel and $\varepsilon_i = 1$ if i is active and $\varepsilon_i = 0$ if i is failing.

The sample space Ω can be written as,

$\Omega = \{(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n); \varepsilon_i \in \{0,1\} \forall i\}$

Let A_k denote the event in which the communication rate is ρk,

$A_k = \{\omega \in \Omega, \ \ \varepsilon_1 + \varepsilon_2 + ... + \varepsilon_n = k\}$

Hence, F : consists of all union of A_k,

The probability measure P can be formulated as a Binomial distribution such that,

$$P(A_k) = \binom{n}{k} \rho^{n-k} (1-\rho)^k; \ \ 0 < \rho < 1$$

4) a) The sample space Ω can be written as,

$\Omega = \{\omega : \omega = (d_0 d_1 ... d_7; d_i \in \{0,1\})\}$

$F = P(\Omega) =$ All the power set $= 2^{\Omega}$

$$P(A) = \frac{\text{Cardinality of A}}{\text{Cardinality of } \Omega} = \frac{|A|}{256} \text{ for any event } A \in F$$

b) Consider the following events:

$E_1 =$ No two neighboring digits are the same

$= \{01010101, 10101010\}$

$$P(E_1) = \frac{2}{256} = \frac{1}{128}$$

$E_2 =$ Some cyclic shift of the register contains is equal to 01100110

$= \{01100110, 00110011, 10011001, 11001100\}$

$$P(E_2) = \frac{4}{256} = \frac{1}{64}$$

$E_3 =$ The register contains exactly 4 zeros

$= \{\omega \in \Omega; \mathrm{d}_0 + d_1 + ... + d_7 = 4\}$

$$P(E_3) = \frac{\binom{8}{4}}{256} = \frac{70}{256} = \frac{35}{128}$$

E_4 = There is a run of at least 6 consecutive ones

$$= \{01111110, 00111111, 11111100, 11111110, 01111111, 11111111, 10111111, 11111101\}$$

$$P(E_4) = \frac{8}{256} = \frac{1}{32}$$

c) $P(E_1 / E_3) = \dfrac{P(E_1 \cap E_3)}{P(E_3)} = \dfrac{P(E_1)}{P(E_3)} = \dfrac{1}{35}$

$$P(E_2 / E_3) = \frac{P(E_2 \cap E_3)}{P(E_3)} = \frac{P(E_2)}{P(E_3)} = \frac{2}{35}$$

5) The sample space Ω can be written as,

$$\Omega = \{w : w = (w_1, w_2, \ldots); w_i \in \{H, T\}\ \forall i\}$$

Let E_i be the event i^{th} flip comes up heads

$$E_i = \{w \in \Omega; w_i = H\},\ P(E_i) = p$$

a) Heads are got for the first time exactly on the n[th] flip = $\{ X(w) = n \}$

$$\{X = n\} = \{w = (\underbrace{T, T, T, \ldots,}_{n-1 \text{times}} H, \ldots)\}$$

$$= E_1^c \cap E_2^c \cap \ldots. \cap E_{n-1}^c \cap E_n$$

$$P(\{X = n\}) = P(E_1^c \cap E_2^c \cap \ldots \cap E_n)$$

$$= P(E_1^c)P(E_2^c)\ldots P(E_{n-1}^c)P(E_n)$$

$$= (1-p)^{n-1} p$$

b) Heads are got for the first time at time 6 or later.

$$\{X \geq 6\} = \{w = (T, T, T, T, T, \underbrace{\ldots}_{\text{can be H or T}}\}$$

$$= E_1^c \cap E_2^c \cap \ldots. \cap E_5^c$$

$$P(\{X \geq 6\}) = P(E_1^c \cap E_2^c \cap \ldots \cap E_5^c)$$

$$= P(E_1^c)P(E_2^c)\ldots P(E_5^c)$$

$$= (1-p)^5$$

c) Heads are got for the first time at the least by time 5 = $\{X \leq 5\}$

$$P(\{X \leq 5\}) = P(\{X \geq 6\}^c)$$

$$= 1 - P(\{X \geq 6\})$$

$$= 1 - (1-p)^5$$