Faculty of Engineering
EM 509 – Stochastic Processes
Answer Sheet
Special Classes of Stochastic Processes

1)

a) An iid Bernoulli process with probability of success *p*.

The joint distribution of any sampling times can be expressed as the product of the first order marginal distribution. Therefore, this stochastic process is independent and identically distributed. Thus, **it is SSS and WSS.**

b) A process $X_n = A\sin(n\omega_0 + \phi)$, where *A* is a Gaussian random variable with mean μ_A and variance σ_A^2

$$
\begin{aligned}
\mu_X(n) &= E(X_n) \\
&= E(A\sin(n\omega_0 + \phi)) \\
&= E(A)\sin(n\omega_0 + \phi) \\
&= \mu_A \sin(n\omega_0 + \phi)
\end{aligned}
$$

The mean of X_n is depend on time *n* **. Thus, this process is not WSS and SSS.**

c) A process $X(n) = A\cos(n\omega_0) + B\sin(n\omega_0)$ where *A* and *B* are uncorrelated zero mean random variables with variance σ^2 and ω_0 is a constant.

The mean of $X(n)$:

$$
\begin{aligned}
\mu_X(n) &= E(A\cos(n\omega_0) + B\sin(n\omega_0)) \\
&= E(A)\cos(n\omega_0) + E(B)\sin(n\omega_0) \\
&= 0.\cos(n\omega_0) + 0.\sin(n\omega_0) \\
&= 0
\end{aligned}
$$

Therefore mean of $X(n)$ does not depend on time.

The autocorrelation of $X(n)$:

$$
\begin{aligned}
R_{XX}(n,m) &= E(X_n X_m) \\
&= E((A\cos(n\omega_0) + B\sin(n\omega_0))(A\cos(m\omega_0) + B\sin(m\omega_0)) \\
&= E(A^2)\cos(n\omega_0)\cos(m\omega_0) + E(B^2)\sin(n\omega_0)\sin(m\omega_0) + E(AB)\sin(\omega_0(n+m)) \\
&= \sigma^2\cos(n\omega_0)\cos(m\omega_0) + \sigma^2\sin(n\omega_0)\sin(m\omega_0) + 0.\sin(\omega_0(n+m)) \\
&= \sigma^2\cos(\omega_0(n-m))
\end{aligned}
$$

The autocorrelation depends on the time difference. Therefore, **the random process $X(n)$ is WSS.** The sequence of $X(n)$ depend on ω_0. Thus, **the random process $X(n)$ is not always SSS.**

d) A process $Y(n) = X(n) - X(n-1)$ where $X(n)$ is an iid Bernoulli process with probability of success *p*.

Mean:

$$
\begin{aligned}
\mu_Y(n) &= E[X(n) - X(n-1)] \\
&= E[X(n)] - E[X(n-1)] \\
&= p - p \\
&= 0
\end{aligned}
$$

Variance:

$$
\begin{aligned}
\sigma_Y(n) &= E\left[(X(n) - X(n-1))^2\right] - (E[X(n) - X(n-1)])^2 \\
&= E\left[(X(n))^2\right] - 2E[X(n)X(n-1)] + E\left[(X(n-1))^2\right] \\
&= p(1-p) + p^2 - 2p^2 + p(1-p) + p^2 \\
&= 2p(1-p)
\end{aligned}
$$

Autocovariance:

$$C_{YY}(n,m) = E(Y_n Y_m)$$
$$= E[(X(n) - X(n-1))(X(m) - X(m-1))]$$
$$= E[X(n)X(m)] - E[X(n)X(m-1)] - E[X(n-1)X(m)] + E[X(n-1)X(m-1)]$$

if $n \neq m$

$$C_{YY}(n,m) = E(X(n))E(X(m)) - E(X(n))E(X(m-1)) - E(X(n-1))E(X(m)) + E(X(n-1))E(X(m-1))$$
$$= \mu^2 - \mu^2 - \mu^2 + \mu^2$$
$$= 0$$

if $n = m$

$$C_{YY}(n,m) = E((X(n))^2) - 2\ E[X(n)X(n-1)] + E[(X(n-1))^2]$$
$$= p(1-p) + p^2 - 2p^2 + p(1-p) + p^2$$
$$= 2p(1-p)$$

$$\text{Hence, } C_{YY}(n,m) = \begin{cases} 2\sigma^2 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Autocorrelation:

$$R_{YY}(n,m) = C_{YY}(n,m) + \mu_Y(n)\mu(m)$$
$$= C_{YY}(n,m)$$

The mean and autocorrelation do not depend on time and the process of $Y(n)$ is iid .Therefore, **the random process $Y(n)$ is WSS and SSS.**

2) Consider a random process, $X(t) = U\cos(\omega_0 t) + V\sin(\omega_0 t)$, where ω_0 is a constant and U, V are random variables.

   i) The mean of $X(t)$

   $$\mu_X(t) = E(U\cos(\omega_0 t) + V\sin(\omega_0 t))$$
   $$= E(U)\cos(\omega_0 t) + E(V)\sin(\omega_0 t)$$

   The only way to make the mean constant is to have $E(U) = E(V) = 0$

   ii) The autocorrelation of $X(t)$

$$
\begin{aligned}
R_{XX}(t_1,t_2) &= E(X(t_1)X(t_2)) \\
&= E[(U\cos(\omega_0 t_1)+V\sin(\omega_0 t_1))(U\cos(\omega_0 t_2)+V\sin(\omega_0 t_2))] \\
&= E(U^2)\cos(\omega_0 t_1)\cos(\omega_0 t_2)+E(UV)\cos(\omega_0 t_1)\sin(\omega_0 t_2)+ \\
&\quad E(VU)\sin(\omega_0 t_1)\cos(\omega_0 t_2)+E(V^2)\sin(\omega_0 t_1)\sin(\omega_0 t_2)
\end{aligned}
$$

If U,V are uncorrelated with equal variance.

$(E(UV)=E(VU)=0, E(U^2)=E(V^2)=\sigma^2)$

$R_{XX}(t_1,t_2)=\sigma^2(\cos(\omega_0(t_1-t_2)))$

Then the autocorrelation depends on time difference. Therefore, this process is WSS, if U,V are uncorrelated with equal variance.

If the process is WSS, then the autocorrelation depends on time difference.

Let , $\tau = t_1 - t_2$

$$
\begin{aligned}
R_{XX}(\tau) &= E[X(0)X(\tau)] \\
&= E[U(U\cos(\omega_0\tau)+V\sin(\omega_0\tau))] \\
&= E(U^2)\cos(\omega_0\tau)+E(UV)\sin(\omega_0\tau)
\end{aligned}
$$

Hence,

$R_{XX}(t_1,t_2)=R_{XX}(\tau)$

$E(U^2)\cos(\omega_0 t_1)\cos(\omega_0 t_2)+E(UV)\cos(\omega_0 t_1)\sin(\omega_0 t_2)+E(VU)\sin(\omega_0 t_1)\cos(\omega_0 t_2)+$
$E(V^2)\sin(\omega_0 t_1)\sin(\omega_0 t_2)=E(U^2)\cos(\omega_0\tau)+E(UV)\sin(\omega_0\tau)$

$E(U^2)\cos(\omega_0 t_1)\cos(\omega_0 t_2)+E(UV)\cos(\omega_0 t_1)\sin(\omega_0 t_2)+E(VU)\sin(\omega_0 t_1)\cos(\omega_0 t_2)+$
$E(V^2)\sin(\omega_0 t_1)\sin(\omega_0 t_2)=E(U^2)\cos(\omega_0(t_1-t_2))+E(UV)\sin(\omega_0(t_1-t_2))$

$\sin(\omega_0 t_1)\sin(\omega_0 t_2)(E(V^2)-E(U^2))+2E(UV)\cos(\omega_0 t_1)\sin(\omega_0 t_2)=0$

The above condition is true for any t_1,t_2, further $E(U^2)=E(V^2)$ and $E(UV)=0$,this implies that U,V are uncorrelated with equal variance.