Scribe Notes for ”Parameterized Complexity of Problems in Coalition Resource Games”

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1 Overview

Coalition formation is a key topic in multi-agent systems. Coalitions enable agents to achieve goals that they may not have been able to achieve on their own.
Some previous papers have shown problems in coalition games to be computationally hard. Woolbridge and Dunne [10] studied the computational complexity of several natural decision problems in Coalition Resource Games (CRGs) - games in which each agent is endowed with a set of resources and coalitions can bring about a set of goals if they are collectively endowed with the necessary amount of resources. However such hardness results consider the entire input as one and we lose out on using structural information hidden in input. In case of coalition resource games this bundles together several distinct elements of the input e.g. the agent set, the goal set, the resources etc. Shrot, Aumann and Kraus [6] examine the complexity of coalition formation problems in the CRG model as function of distinct input elements using the theory of Parameterized Complexity. Their refined analysis shows that not all parts of input act equal - some instances of the problem are actually tractable.

2 Coalitions

In multiagent systems (MAS), where each agent has limited resources, the formation of coalitions of agents is a very powerful tool [8][7]. Coalitions enable agents to accomplish goals they may not have been able to accomplish independently. As such, understanding and predicting the dynamics of coalitions formation, e.g. which coalitions are more beneficial and/or more likely to emerge, is a question of considerable interest in multi-agent settings. Unfortunately, a range of previous studies have shown that many of these problems are computationally complex [10][9]. Nonetheless, as noted by Garey and Johnson [5], hardness results such as NP-completeness should merely constitute the beginning of the research. NP-hardness indicates that a general solution for all instances of the problem most probably does not exist. Still, efficient solutions for important sub-classes may well exist.

3 Formal model of Coalition Resource Games

The framework we use to model coalitions is the CRG model introduced in [10], defined as follows. The model contains a non-empty, finite set $Ag = \{a_1, \ldots, a_n\}$ of *agents*. A *coalition*, typically denoted by C, is simply a set of agents, i.e. a subset of Ag. The *grand coalition* is the set of all agents, Ag. There is also a finite set of *goals* G. Each agent $i \in Ag$ is associated with a subset G_i of the goals. Agent i is *satisfied* if at least one member of G_i is achieved, and unsatisfied otherwise. We also make the assumption that agent i is indifferent among members of G_i

Achieving the goals requires the expenditure of *resources*, drawn from the total set of resources R. Achieving different goals may require different quantities of each resource. The quantity **req**(g, r) denotes the amount of resource r required to achieve goal g. It is assumed that **req**(g,r) is a natural number. Each agent is *endowed* certain amounts of some or all of the resources. The quantity **en**(i,r) denotes the amount of resource r endowed to agent i. Again, it is assumed that **en**(i,r) is a natural number or zero.

Formally we have that a *Coalition Resource Game* Γ is a (n+5)-tuple given by

$$\Gamma = \langle Ag, G, R, G_1, G_2, \ldots, G_n, \mathbf{en}, \mathbf{req} \rangle$$

where :

- $Ag = \{a_1, a_2, \ldots, a_n\}$ is a set of agents
- $G = \{g_1, g_2, \ldots, g_m\}$ is a set of possible goals
- $R = \{r_1, r_2, \ldots, r_t\}$ is a set of resources
- For each $i \in$ Ag, $G_i \subseteq G$ is a set of goals such that any of the goals in G_i would satisfy i but i is indifferent between members of G_i
- **en** : Ag $\times$ R $\rightarrow \mathbb{N} \cup \{0\}$ is an endowment function
- **req** : G $\times$ R $\rightarrow \mathbb{N} \cup \{0\}$ is a requirement function

The endowment function **en** extends to coalitions by summing up endowments of its members as follows

$$\mathbf{en}(C, r) = \sum_{i \in C} \mathbf{en}(i, r) \quad \forall r \in R$$

The requirement function **req** extends to sets of goals by summing up requirements of its members as follows

$$\mathbf{req}(G', r) = \sum_{g \in G'} \mathbf{req}(g, r) \quad \forall r \in R$$

A set of goals G' *satisfies* agent i if $G_i \cap G \neq \emptyset$ and satisfies coalition C if it satisfies every member of C. A set of goals G' is *feasible* for coalition C if that coalition is endowed with sufficient resources to achieve all goals in G' i.e. for all $r \in R$ we have $\mathbf{req}(G', r) \leq \mathbf{en}(G', r)$. Finally we say that a coalition C is *successful* if there exists a non-empty set of goals G' that satisfies C and is feasible for it. In general, we use the following notation $succ(C) = \{G' \mid G' \subseteq G\ ;\ G' \neq \emptyset$ and G' both satisfies C and is feasible for it$\}$

The CRG model is a simple model but it models many real-world situations like the virtual organizations problem [1]. A virtual organization is a temporary alliance of organizations that come together to share skills and resources in order to better respond to business opportunities. CRG model can also model voting domains.

4 Parameterized Complexity

We now provide a brief introduction to the key relevant concepts from the theory of parameterized complexity. The definitions in this section are taken from [4] and [2].

The core idea of parameterized complexity is to single out a specific part of the input as the parameter and ask whether the problem admits an algorithm that is efficient in all but the parameter. In most cases the parameter is simply one of the elements of the input (e.g. the size of the goal set), but it can actually be any computable function of the input:

Definition 1. *Let Σ be a finite alphabet.*

1. *A* ***parametrization*** *of Σ^* is a* mapping $\kappa : \Sigma^* \to \mathbb{N}$ *that is* polynomial time computable.
2. *A* ***parameterized problem*** *(over Σ) is a pair (Q, κ) consisting of a set $Q \subseteq \Sigma^*$ of strings over Σ and a parameterization κ of Σ^**

As stated, given a parameterized problem we seek an algorithm that is efficient in all but the parameter. This is captured by the notion of *fixed parameter tractability*, as follows:

Definition 2. *A parameterized problem (Q, κ) is fixed. parameter tractable (FPT) if there exist an algorithm $\mathbb{A}$, a constant α, and a computable function f, such that $\mathbb{A}$ decides Q in time $f(\kappa(x))|x|^\alpha$*

Thus, while the fixed-parameter notion allows inefficiency in the parameter $\kappa(x)$, by mean of the function f, it requires polynomial complexity in all the rest of the input. In particular, a problem that is FPT is tractable for any bounded parameter value.

While the core aim of parameterized complexity is to identify problems that are fixed-parameter tractable, it has also developed an extensive complexity theory, allowing to prove hardness results, e.g. that certain problems are (most probably) not FPT . To this end, several parameterized complexity classes have been defined. Two of these classes are the class W[1] and the class para-NP. We will formally define these classes shortly, but the important point to know is that there is strong evidence to believe that both classes are not contained in FPT (much like NP is probably not contained in P). Thus, W[1]-hard and para-NP-hard problems are most probably not fixed-parameter tractable.

The class W[1] can be defined by its core complete problem, defined as follows :
SHORT NONDETERMINISTIC TURING MACHINE COMPUTATION
Instance : A single-tape, single-head non-deterministic Turing machine M, a word x and an integer k
Question : Is there a computation of M on input x that reaches the accepting state in at most k steps?
Parameter : k

Note that this definition is analogous to that of NP, with the addition of the parameter k.

Definition 3. *The class W[1] contains all parameterized problems FPT -reducible (defined hereunder) to Short-Nondeterministic-Turing-Machine-Computation.*

The class para-NP is defined as follows

Definition 4. *A parameterized problem (Q, κ) is in para- NP if there exists a non-deterministic Turing machine M, constant α and an arbitrary computable function f, such that for any input x, M decides if $x \in Q$ in time $\leq |x|^\alpha f(\kappa(x))$*

Establishing hardness results most frequently requires reductions. In parameterized complexity, we use FPT -reduction, defined as follows:

Definition 5. *Let $(Q, \kappa)and(Q', \kappa')$ be parameterized problems over the alphabets Σ and Σ' respectively. An FPT-reduction (FPT many.one reduction) from (Q, κ) to (Q', κ') is a mapping $R : \Sigma^* \to (\Sigma')^*$ such that:*

1. *For all $x \in \Sigma^*$ we have $x \in Q \Leftrightarrow R(x) \in Q'$*
2. *R is computable in time $f(\kappa(x))|x|^\alpha$ for some constant α and an arbitrary function f.*
3. *There is a computable function $g : \mathbb{N} \to \mathbb{N}$ such that $\kappa'(R(x)) \leq g(\kappa(x))$ for all $x \in \Sigma^*$*

Point (1) simply states that R is indeed a reduction. Point (2) says that it can be computed in the right amount of time - efficient in all but the parameter. Point (3) states that the parameter of the image is bounded by (a function of) that of the source. This is necessary in order to guarantee that FPT reductions preserve FPT -ness i.e. with this definition we obtain that if (Q, κ) reduces to (Q', κ') and $(Q', \kappa') \in$ FPT then (Q, κ) is also in FPT .

5 Problems related to Coalition Formation

Shrot, Aumann and Kraus [6] considered problems related to coalitions. Specifically they considered the following four problems.

1. SUCCESSFUL COALITION (SC)
   - Instance : A CRG Γ and a coalition C
   - Question : Is C successful ?
2. EXISTENCE OF SUCCESSFUL COALITION OF SIZE K (ESCK)
   - Instance : A CRG Γ and an integer k
   - Question : Does there exist a successful coalition of size exactly k ?
3. MAXIMAL COALITION (MAXC)
   - Instance : A CRG Γ and a coalition C
   - Question : Is every proper superset of C not successful ?
4. MAXIMAL SUCCESSFUL COALITION (MAXSC)
   - Instance : A CRG Γ and a coalition C
   - Question : Is C successful and every proper superset of C not successful ?

The results from [10] and [6] are summarised in the table below

	SC	**ESCK**	**MAXC**	**MAXSC**
	NP-complete	NP-hard	co-NP-complete	D^p-complete
$\|G\|$	FPT	FPT	FPT	FPT
$\|C\|$		W[1]-hard	W[1]-hard	W[1]-hard
$\|R\|$	Para-NP-hard		Para-NP-hard	Para-NP-hard
$\|Ag\| + \|R\|$	FPT		FPT	FPT

6 Conclusions

The study of problems arising in coalitions of agents in multi-agents systems using the parameterized complexity paradigm was initiated by Shrot, Aumann and Kraus [6]. This direction is still unexplored and there are various (classically) computationally hard problems out there waiting to be better analyzed through the rich theory of parameterized complexity.

References

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