# Computational Analyses of the Electoral College: Campaigning Is Hard But Approximately Manageable

Sina Dehghani, Hamed Saleh, Saeed Seddighin, Shang-Hua Teng

## Problem Definition

The Original *Colonel Blotto* Game:

- Two Colonels $\boldsymbol{A}$ and $\boldsymbol{B}$ have $\boldsymbol{m}$ and $\boldsymbol{n}$ troops respectively.
- $\boldsymbol{k}$ simultenous battlefields
- Winner-take-all rule
- All troops have the same strength.
- *Borel* (1921) $\rightarrow$ *Ahmadinejad et al.* (2016)

The Colonel Blotto Game with *Multifaceted Resources*:

- Troops do not necessarily have the same strength.
- Their strength also varies in different battlefields.
- Arbitrary utility functions for each battlefield.
- Constant-sum, monotone, non-negative

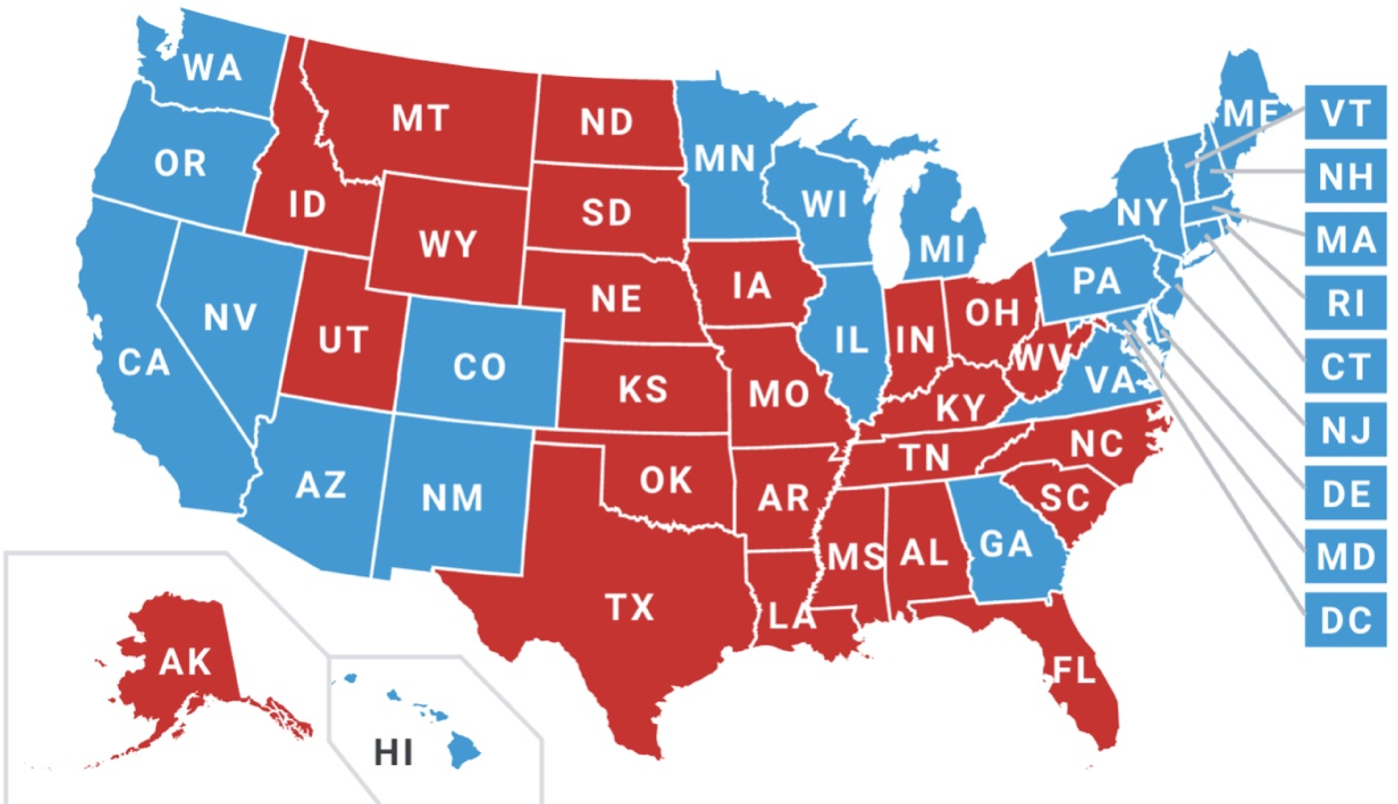
Goal:

- Finding Nash Equilibria of the game in polynomial time
- Constant-sum games: Nash Equilibrium $\equiv$ Maxmin Strategy

## U.S. Presidential Election

A

B

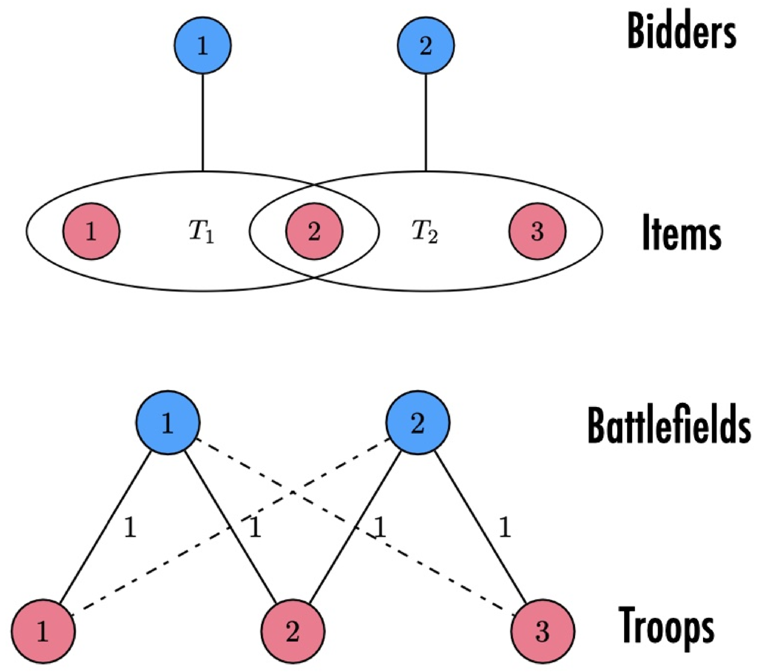



$\boldsymbol{k} = \mathbf{50}$

Troops may include money, staff, candidates' time, etc.

## Biceriteria Approximation

- Hard to approximate the best response within $\sqrt{\boldsymbol{n}}$ factor.
- *Welfare Maximization for Single-minded Bidders*




- Biceriteria approximation to the rescue.
- Meaningful approximation by sacrificing more resources.

**Definition**

$(\alpha, \beta)$-*best-response*:

- Using $\alpha$ copies of each troop is allowed.
- Payoff is at least $\mathbf{1}/\beta$ fraction of optimal.

**Definition**

$(\alpha, \delta)$-*maxmin strategy*:

- Using $\alpha$ copies of each troop is allowed.
- Minimum payoff is at least that of optimal strategy minus $\delta$.

## Approximate Best Response

**Theorem**

For any $\epsilon > \mathbf{0}$, there is a polynomial algorithm for finding an $(\boldsymbol{O}(\frac{\ln 1/\epsilon}{\epsilon}), \frac{1}{1-\epsilon})$-best-response.

- Plug into the reduction as a best response oracle.
- $(\boldsymbol{O}(\frac{\ln 1/\epsilon}{\epsilon}), \mathbf{2}\epsilon)$-maxmin strategy.
- Almost optimal by allowoing $\boldsymbol{O}(\mathbf{1})$ copies of each troop.

It is possible to significantly improve the approximation if troops are homogenous wrt to different battlefields:

**Theorem**

For any $\epsilon > \mathbf{0}$, there is a polynomial algorithm for finding an $(\mathbf{1} + \epsilon, \mathbf{1})$-best-response in the homogenous setting.

## Reduction to $(\alpha, \beta)$-best-response

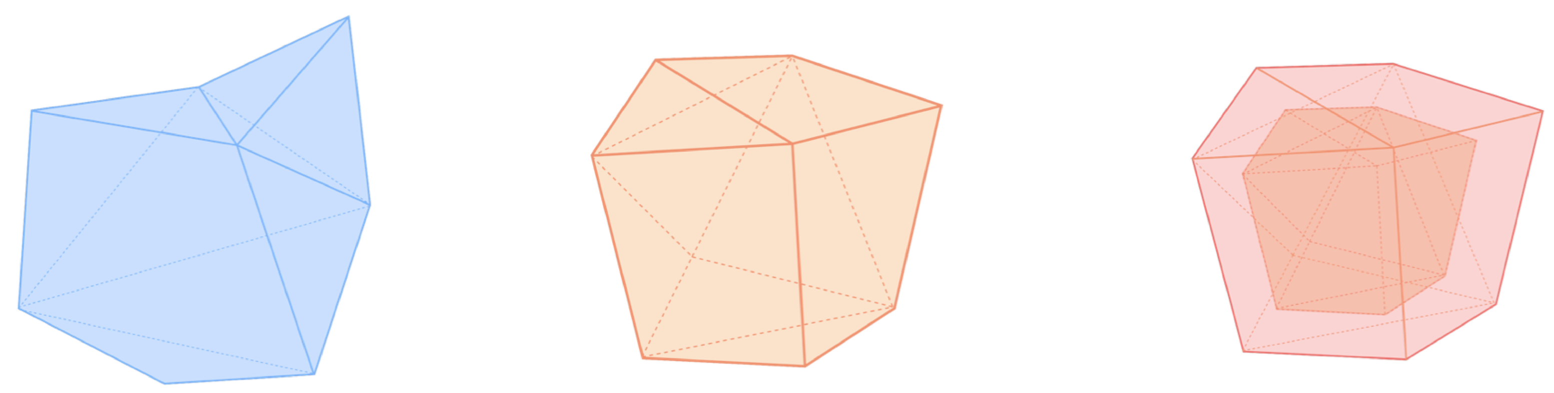
Model the problem as a Linear Program:

$$\begin{array}{lll} \max. & U & \\ \text{s.t.} & \hat{x} \in S(\mathbf{A}) & \textbf{Membership constraints} \\ & \mu^{\mathbf{A}}(\hat{x}, \hat{y}) \geq U, \qquad \forall \hat{y} \in S(\mathbf{B}) & \textbf{Payoff constraints} \end{array}$$

- $\boldsymbol{S}(\boldsymbol{A})$: Set of all feasible (mixed) strategies for colonel $\boldsymbol{A}$.
- $\boldsymbol{\mu}^{\boldsymbol{A}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$: The payoff of colonel $\boldsymbol{A}$ for strategies $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$.
- $\hat{\boldsymbol{x}}_{(\boldsymbol{s},\boldsymbol{b})}$: The probability of assigning a subset of troops with total strength $\boldsymbol{s}$ to battlefield $\boldsymbol{b}$.

Membership Constraints:

- Checking whether a point $\hat{\boldsymbol{x}}$ is inside polytope $\boldsymbol{S}(\boldsymbol{A})$.
- Find a separating hyperplane?
- Requires exact best reponse.
- But, we can approximately check the membership.
- $\boldsymbol{S}'(\boldsymbol{A})$: the set of points that our approximate algorithm admits.
- Not necessarily convex!
- Good approximation of $\boldsymbol{S}(\boldsymbol{A})$.



$S'(A)$ $S(A)$ $S(A)/\beta$

**Theorem**

Given a $(\alpha, \beta)$-best-response oracle, there is a polynomial algorithm for finding an $(\alpha, \mathbf{2} - \mathbf{2}/\beta)$-maxmin strategy.

## Related Work

- From Duels to Battlefields: Computing Equilibria of Blotto and Other Games. [AAAI'16]
- Spatio-Temporal Games Beyond One Dimension. [EC'18]