Fair Allocation of indivisible goods with externalities

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Externalities
Fair division problem

There are objects to be distributed among agent, where each agent gains a utility, when an object is allocated to her.

\[ V_i(\{b\}) \]
Fair division problem

With externalities

There are objects to be distributed among agents,
where each agent gains a utility,
when an object is allocated to anyone.

items allocated to other agents is important for each agent.
Fair division problem

With externalities

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Fair division problem

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Model
General Externalities Model

Suppose set $S$ is allocated to agent $j$,
then agent $i$ gains utility of

$$V_{j,i}(S)$$
General Externalities Model

Suppose set \( S \) is allocated to agent \( j \),

then agent \( i \) gains utility of

\[
V_{j,i}(S) = \sum_{b \in S} V_{j,i}({\{b\}})
\]

Suppose the valuations are additive.
Network Externalities Model

Modeling the externalities based on an influence graph
Network Externalities Model

The utility of each agent is based on the edge weights,

\[ V_{j,i}(S) = \sum_{b \in S} V_{i}([{b}]) \cdot w_{j,i} \]
Network Externalities Model

The weights of the edges are normalized,

$$\sum_j w_{j,i} = 1$$

Modeling the externalities based on an influence graph
Normalized weights

Why is it ok to normalize the weights?

• We can scale the weights and define a fairness criteria independent of the absolute value of the weights.
Normalized weights

What do normalized weights mean?

- Normalized weights could be interpreted as the probability that agent $j$ borrows his allocated items to agent $i$. 
Fairness Criteria
Common Criteria

The most common criteria could be extended for the case with externalities, namely

- Proportionality
- Envy-freeness
- Maximin Share
Extended Proportionality

Branzei et al. (2013)

Consider the maximum utility agent $i$ gains by allocating each item to the right agent,

$$
\hat{V}_i = \sum_{b \in M} \max_{j \in N} V_{j,i}(\{b\})
$$
**Extended Proportionality**

Branzei et al. (2013)

An allocation $A$ is extended-proportional if for each we have

$$U_i(A) \geq \frac{\hat{V}_i}{n}$$
Swap envy-freeness

Velez (2011)
Swap envy-freeness

Velez (2011)
Swap envy-freeness

Velez (2011)

An allocation $\mathbf{A}$ is swap envy-free if for every pair of agents $i$ and $j$ we have

$$V_{i,i}(\mathcal{A}_i) + V_{j,i}(\mathcal{A}_j) \geq V_{i,i}(\mathcal{A}_j) + V_{j,i}(\mathcal{A}_i)$$
Swap Stability

Branzei et al. (2013)
Swap Stability

Branzei et al. (2013)
Swap Stability

Branzei et al. (2013)

An allocation $\mathbf{A}$ is swap stable if for every three agents $i$, $j$, and $k$ we have

$$V_{j,i}(\mathcal{A}_j) + V_{k,i}(\mathcal{A}_k) \geq V_{j,i}(\mathcal{A}_k) + V_{k,i}(\mathcal{A}_j)$$
Relationship between criteria

\[ \text{Swap Stability} \implies \text{Extended Proportionality} \implies \text{Swap Envy-freeness} \]
But is extended proportionality the best extension of proportionality?
Average Share

Ghodsi et al. (2018)

Consider the average utility agent $i$ gains by allocating item $b$ to each agent,

$$\overline{V}_i(\{b\}) = \frac{1}{n} \sum_{j \in \mathcal{N}} V_{j,i}(\{b\})$$
Average Share

Ghodsi et al. (2018)

Average Share of agent $i$ equals the sum of these average values for all items,

$$\bar{V}_i = \sum_{b \in M} \sum_{j \in N} V_{j,i}(\{b\})$$
Average Share

Ghodsi et al. (2018)

An allocation $A$ is average if for each agent we have

$$U_i(A) \geq \bar{V}_i$$
Average Share vs Extended Proportionality

It is easy to observe that in network externalities model, we have the following:

\[
\frac{\hat{V}_i}{n} = \frac{V_i(M) \cdot (\max_j w_{j,i})}{n}
\]

\[
\bar{V}_i = \frac{V_i(M) \cdot (\sum_j w_{j,i})}{n}
\]
Average Share vs Extended Proportionality

Average Share is more sensitive to externalities in comparison to Extended Proportionality.

\[ \hat{V}_i / n = V_i(M) \cdot (\max_j w_{j,i}) / n \]

\[ \bar{V}_i = V_i(M) \cdot (\sum_j w_{j,i}) / n \]
Relationship between criteria

- **Average Share** implies **Extended Proportionality**
- **Swap Stability** implies **Swap Envy-freeness**
Extended Maximin Share

Ghodsi et al. (2018)

We can utilize the notion of cut and choose to find a suitable fairness criterion to capture externalities in fair division of indivisible items.
Extended Maximin Share

Ghodsi et al. (2018)

Cut and choose is consisted of two parts:

1. Division
2. Allocation
Extended Maximin Share

Ghodsi et al. (2018)

1. Division:
   Similar to **Maximin share**, we ask agent $i$ to divide items into $n$ bundles in a balanced way.

2. Allocation

   Note that the valuations is from the point of view of agent $i$. 
Extended Maximin Share

Ghodsi et al. (2018)

1. Division

2. Allocation:
   
   An adversary allocates the bundles to agents in a way that the utility of agent $i$ minimizes.
Extended Maximin Share

Ghodsi et al. (2018)

1. Division

2. Allocation:

   An adversary allocates the bundles to agents in a way that the utility of agent $i$ minimizes.

We call this minimized utility $\text{EMMS}_i$. 
Extended Maximin Share

Ghodsi et al. (2018)

An allocation $A$ guarantees Extended Maximin Share, if for each agent we have

$$U_i(A) \geq \text{EMMS}_i = \max_{P \in \Pi} U_i(\mathcal{W}_i(P))$$

$$\mathcal{W}_i(P) = \arg \min_{A \in \Omega_P} U_i(A)$$
Relationship between criteria

Extended Maximin Share \(\implies\) Average Share \(\implies\) Extended Proportionality \(\implies\) Swap Stability \(\implies\) Swap Envy-freeness
Computation Aspects of EMMS in Network Externalities model
Computing EMMS

We can observe that computing EMMS is equivalent to the following problem:

Given a set of items $M$ and a sorted vector of weights $w$ in decreasing order, what is the maximum value of this function if agent $i$ partition $M$ into $n$ bundles where vector $x$ is the sorted values of the bundles in increasing order.

$$w \cdot x = \sum_{i=1}^{n} w_i \cdot x_i$$
Computing EMMS

Given a set of items $M$ and a sorted vector of weights $w$ in decreasing order, what is the maximum value of this function if agent $i$ partition $M$ into $n$ bundles where vector $x$ is the sorted values of the bundles in increasing order.

$$w \cdot x = \sum_{i=1}^{n} w_i \cdot x_i$$

This is the utility agent $i$ gains if an adversary allocates the bundles.
The most common partitioning schemes are the special cases of this problem:

1. Maximin partition
   \[ w_1 = 1, w_2 = 0, \ldots, w_n = 0 \]

2. Minimax partition
   \[ w_1 = \frac{1}{n-1}, \ldots, w_{n-1} = \frac{1}{n-1}, w_n = 0 \]

3. Leximin partition
   \[ w_1 = 1 - \epsilon, w_2 = \epsilon - \epsilon^2, \ldots, w_n = \epsilon^{n-1} - \epsilon^n \]
Computing EMMS

The most common partitioning schemes are the special cases of this problem:

It is **NP-hard** to compute the value of EMMS.
Computing **EMMS**

minimax

\[
\begin{array}{ccc}
4 & 4 \\
4 & 4 \\
6 & 6 \\
\end{array}
\]

= 12.5

\[
\begin{array}{ccc}
11 \\
\end{array}
\]

maximin

\[
\begin{array}{ccc}
6 & 4 \\
6 & 4 \\
6 & 4 \\
\end{array}
\]

= 12

\[
\begin{array}{ccc}
4 \\
11 \\
\end{array}
\]
Computing EMMS

minimax

maximin

= 11

= 12
Greedy Approach

A simple greedy algorithm would achieve a 1/2-approximation of the optimum answer.
LPT Algorithm

5 5 4 4 3 3 3 3
LPT Algorithm
LPT Algorithm

4 4 3 3 3 3

5 5
LPT Algorithm
LPT Algorithm
LPT Algorithm
LPT Algorithm
LPT Algorithm
LPT Algorithm
LPT Algorithm

LPT partition

Optimal partition
Computing EMMS

Ghodsi et al. (2018)

Theorem 4.3. The LPT algorithms provides a partition which approximates EMMS by a factor of 1/2.
Fair Allocation in Network Externalities model
Self Reliance

We say an agent $i$ is $\beta$-self-reliant if

$$w_{i,i} \geq \beta$$
Theorem 5.2. If all the agents are $\beta$-self-reliant, then there exists an allocation that guarantees $\beta/2EMMS$. 

Main result
Fair allocation

Ghodsi et al. (2018)

**Corollary 5.6.** If all the agents are β-self-relient, then we can find an allocation that guarantees β/4EMMS.

The algorithm depends on the structure of the optimal partition for each agent which we cannot find, but we can use LPT partition instead.
Thank you!