Fair Allocation of indivisible goods

with externalities

Mohammad Ghodsi, Massoud Seddighin, Hamed Saleh

Presenter: Hamed Saleh

Externalities

Fair division problem

There are objects to be distributed among agent,

where each agent gains a **utility**,

when an object is allocated to her.

$$V_i(\{b\})$$

Fair division problem

With externalities

There are objects to be distributed among agent,

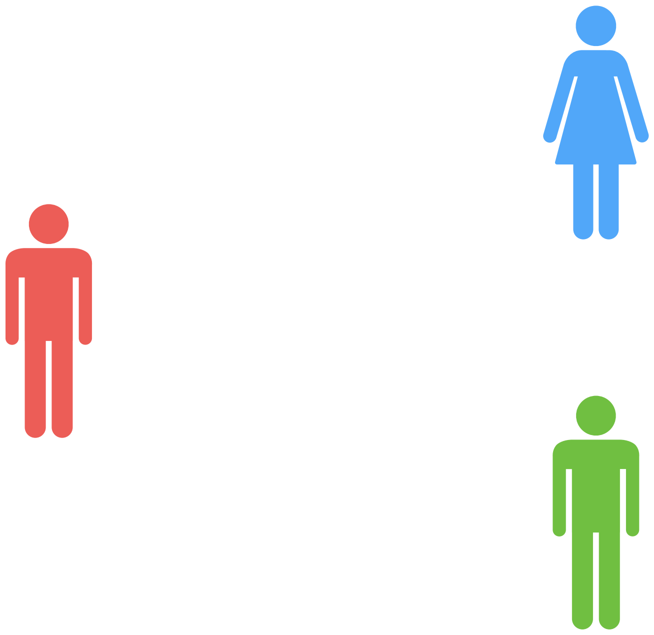
where each agent gains a **utility**,

when an object is allocated to anyone.

items allocated to other agents is important for each agent.

Fair division problem

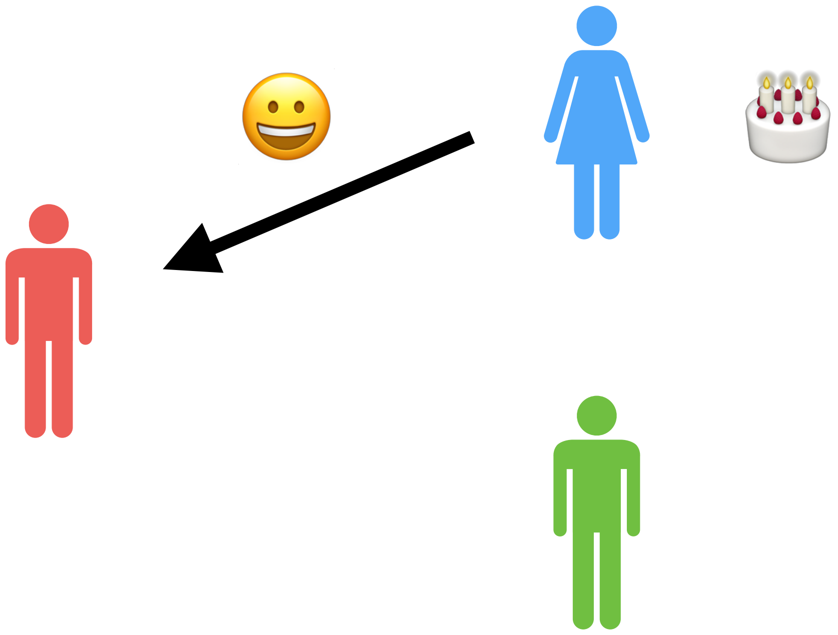
With externalities



items allocated to other agents is important for each agent.

Fair division problem

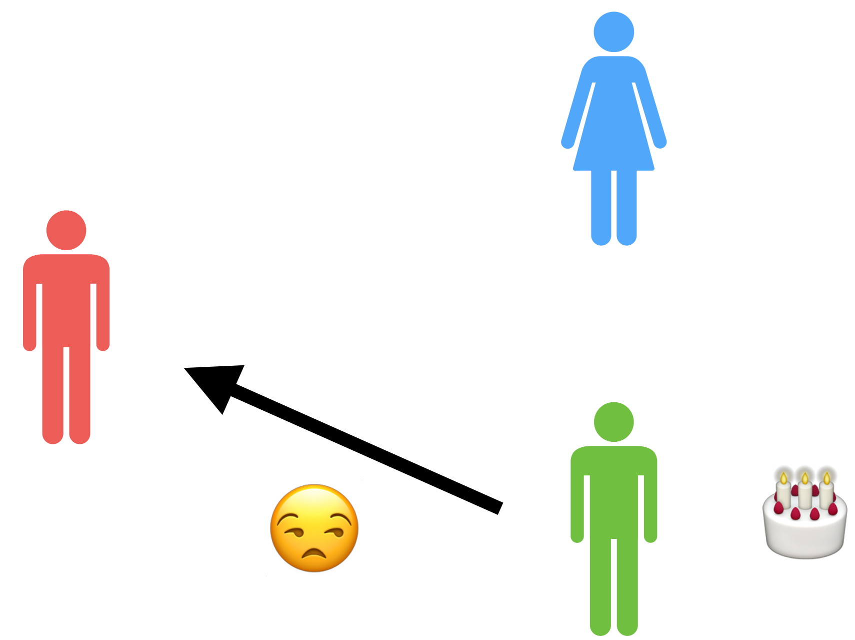
With externalities



items allocated to other agents is important for each agent.

Fair division problem

With externalities



items allocated to other agents is important for each agent.

Model

General Externalities Model

Suppose set **S** is allocated to agent **j**,

then agent **i** gains utility of

$$V_{j,i}(S)$$

General Externalities Model

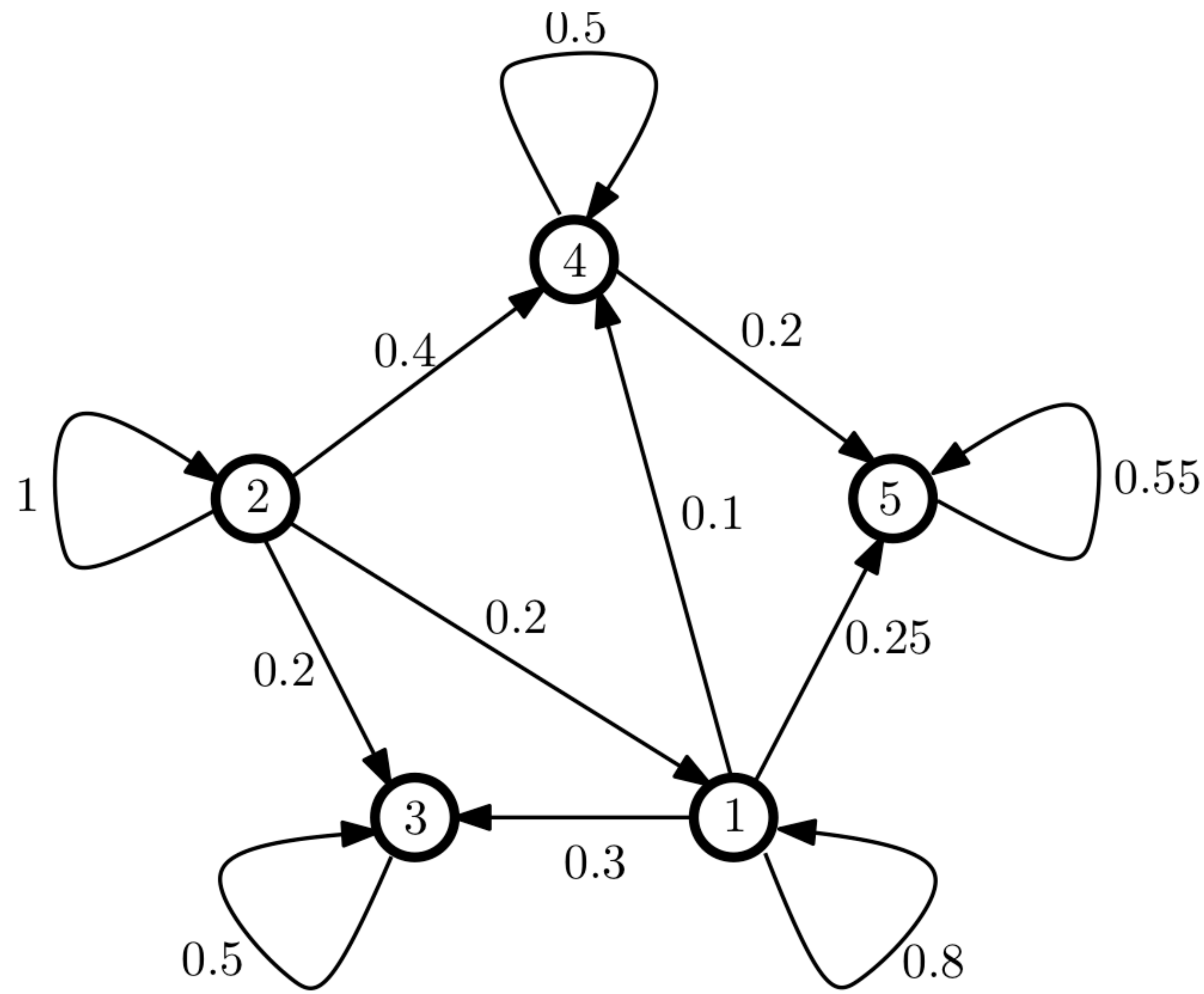
Suppose set **S** is allocated to agent **j**,

then agent **i** gains utility of

$$V_{j,i}(S) = \sum_{b \in S} V_{j,i}(\{b\})$$

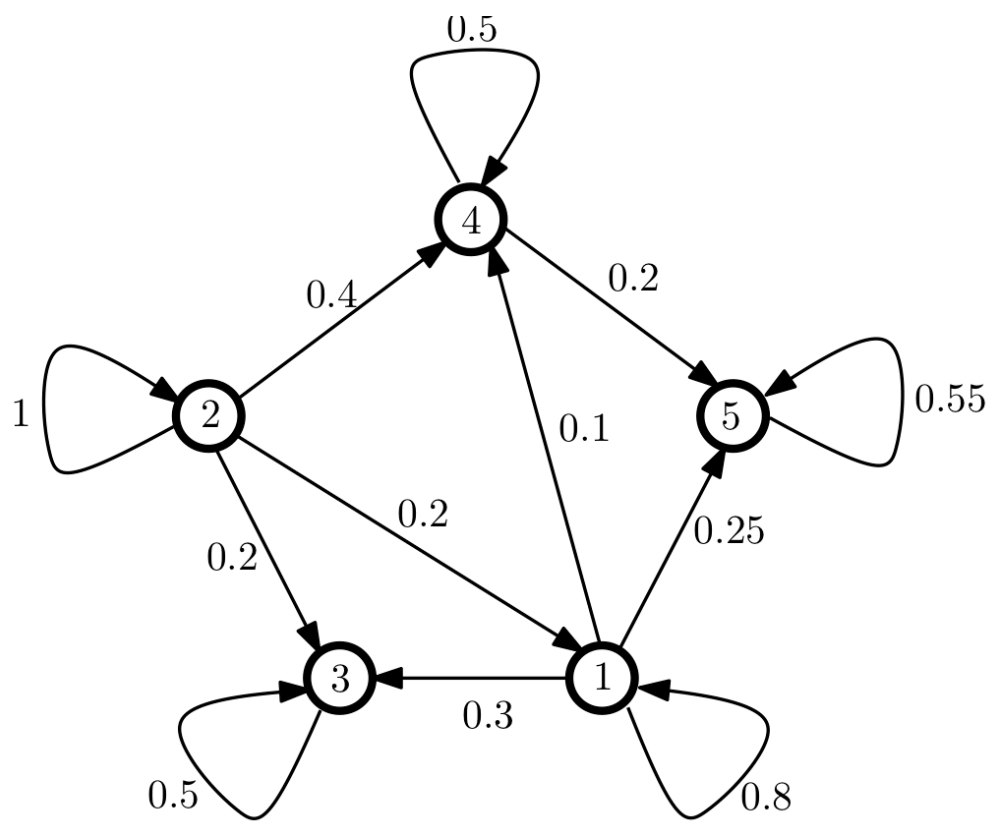
Suppose the valuations are additive

Network Externalities Model



Modeling the externalities based on an influence graph

Network Externalities Model

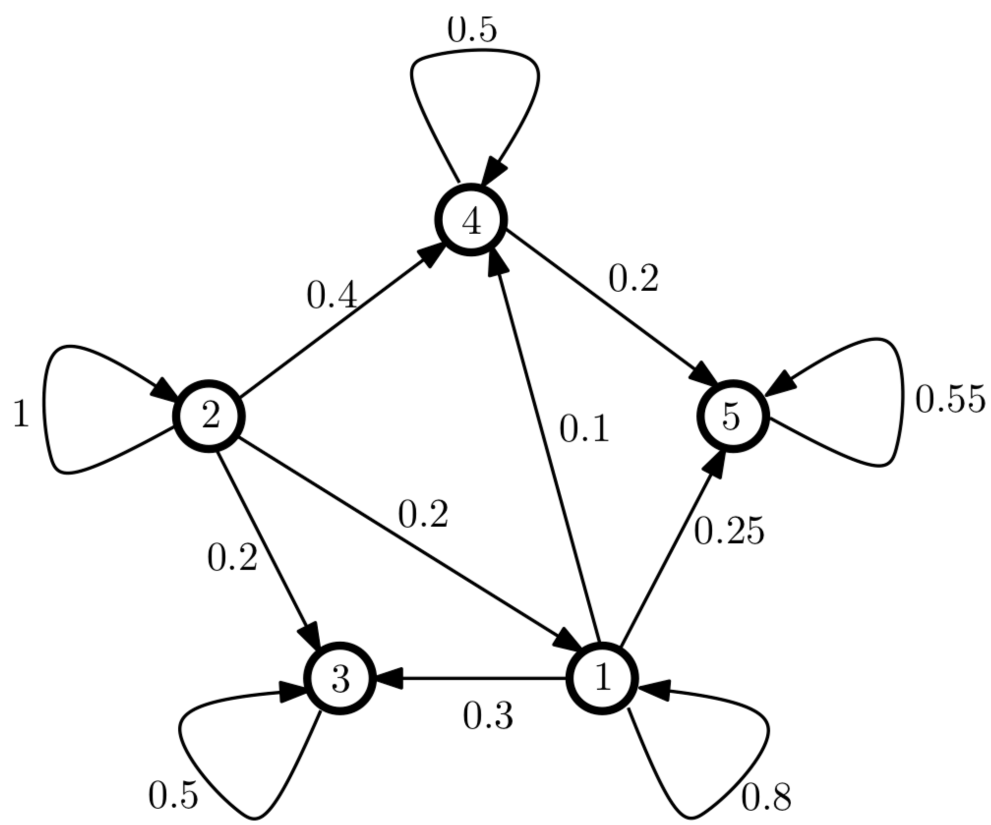


The utility of each agent is based on the edge weights,

$$V_{j,i}(S) = \sum_{b \in S} V_i(\{b\}) \cdot w_{j,i}$$

Modeling the externalities based on an influence graph

Network Externalities Model



The weights of the edges are normalized,

$$\sum_j w_{j,i} = 1$$

Modeling the externalities based on an influence graph

Normalized weights

Why is it ok to normalize the weights?

- We can scale the weights and define a fairness criteria independent of the absolute value of the weights.

Normalized weights

What do normalized weights mean?

- Normalized weights could be interpreted as the **probability** that agent **j** borrows his allocated items to agent **i**.

Fairness Criteria

Common Criteria

The most common criteria could be extended for the case with externalities, namely

- **Proportionality**
- **Envy-freeness**
- **Maximin Share**

Extended Proportionality

Branzei et al. (2013)

Consider the maximum utility agent **i** gains by allocating each item to the right agent,

$$\hat{V}_i = \sum_{b \in \mathcal{M}} \max_{j \in \mathcal{N}} V_{j,i}(\{b\})$$

Extended Proportionality

Branzei et al. (2013)

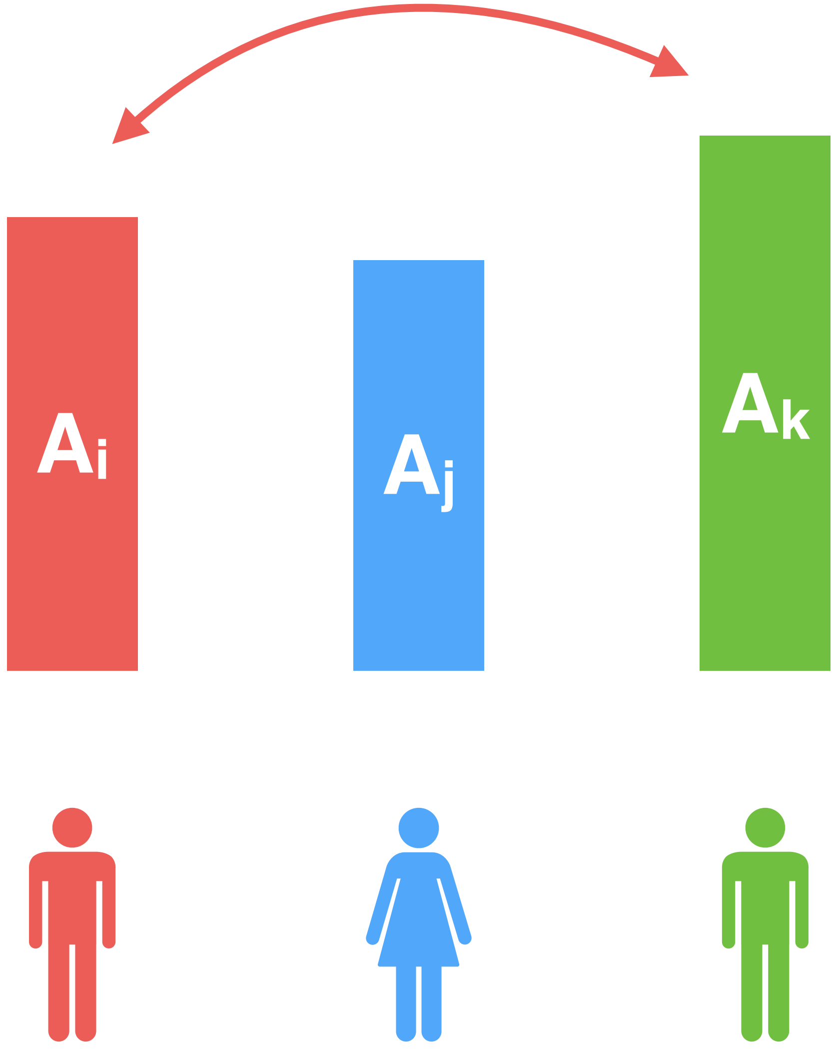
An allocation **A** is extended-proportional

if for each we have

$$U_i(\mathcal{A}) \geq \frac{\hat{V}_i}{n}$$

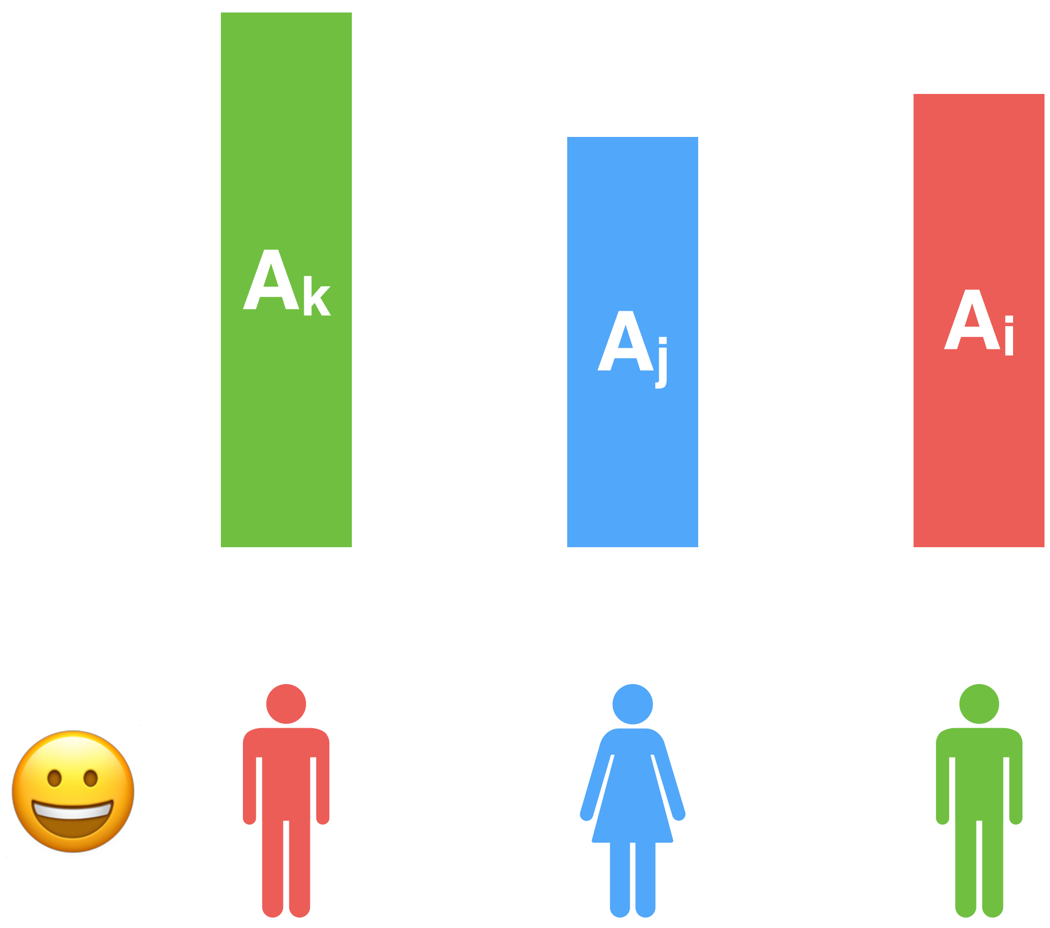
Swap envy-freeness

Velez (2011)



Swap envy-freeness

Velez (2011)



Swap envy-freeness

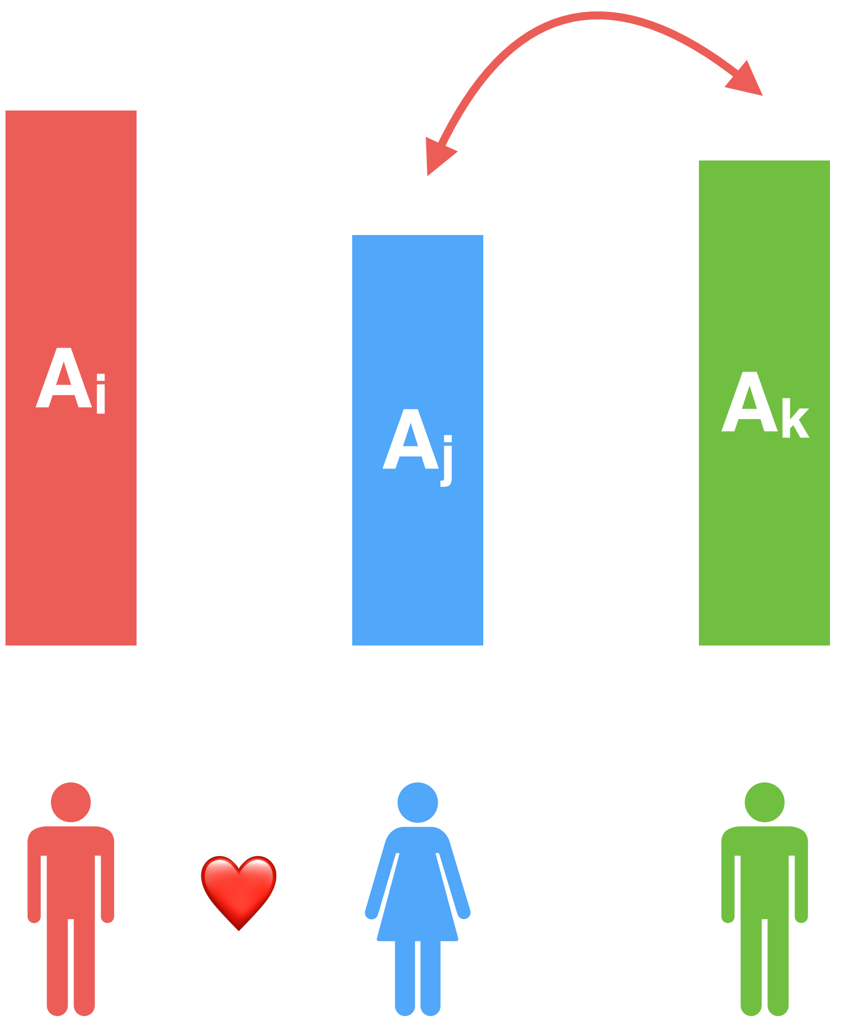
Velez (2011)

An allocation **A** is swap envy-free if for every pair of agents **i** and **j** we have

$$V_{i,i}(\mathcal{A}_i) + V_{j,i}(\mathcal{A}_j) \geq V_{i,i}(\mathcal{A}_j) + V_{j,i}(\mathcal{A}_i)$$

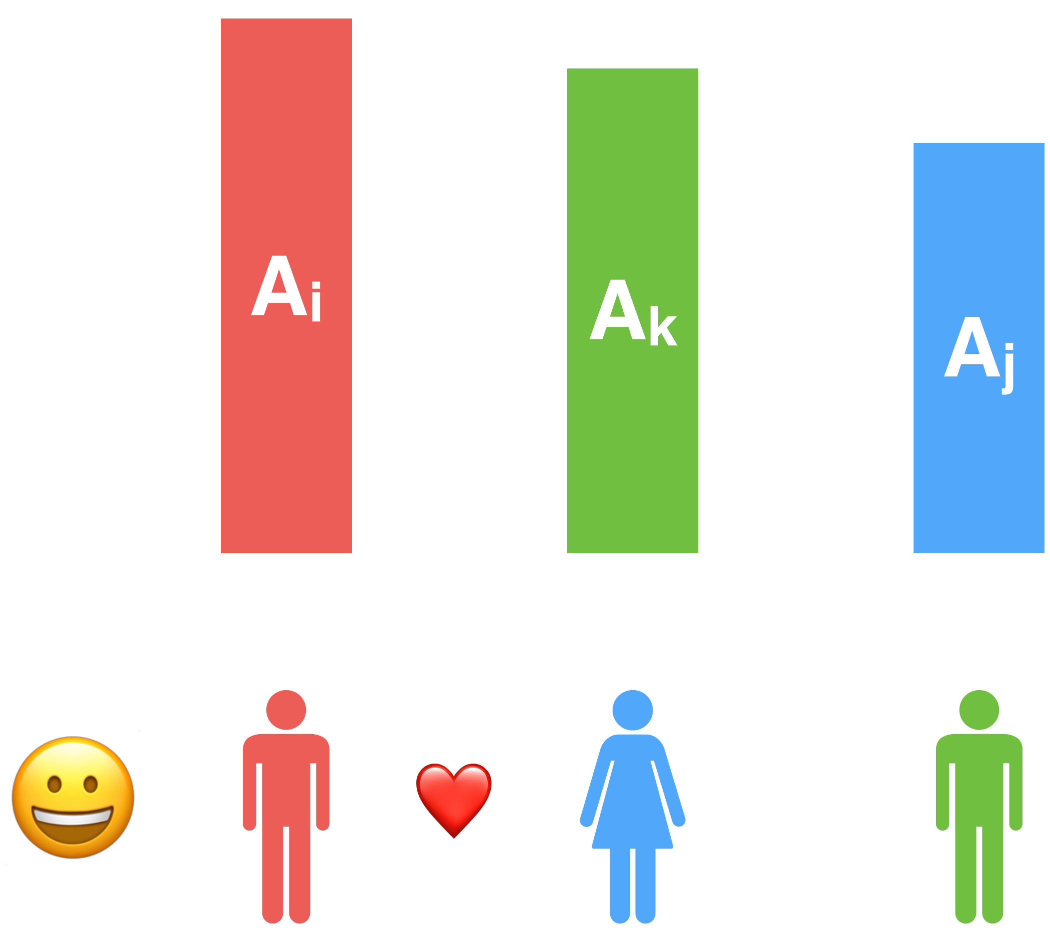
Swap Stability

Branzei et al. (2013)



Swap Stability

Branzei et al. (2013)



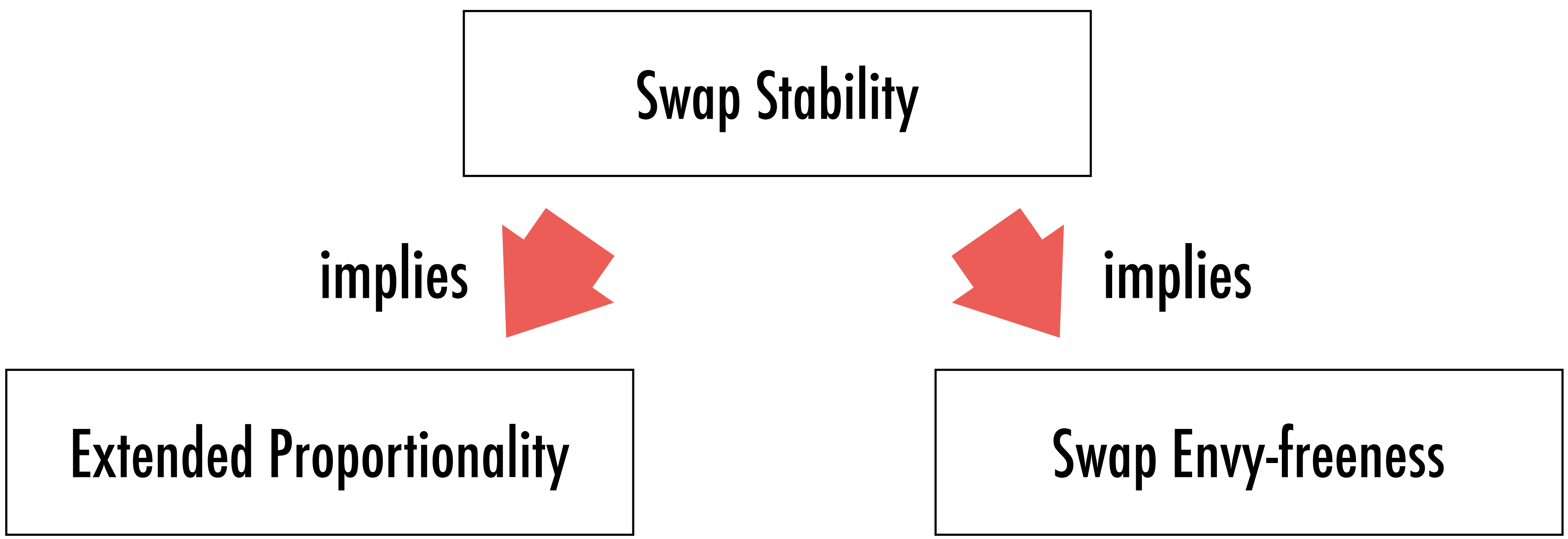
Swap Stability

Branzei et al. (2013)

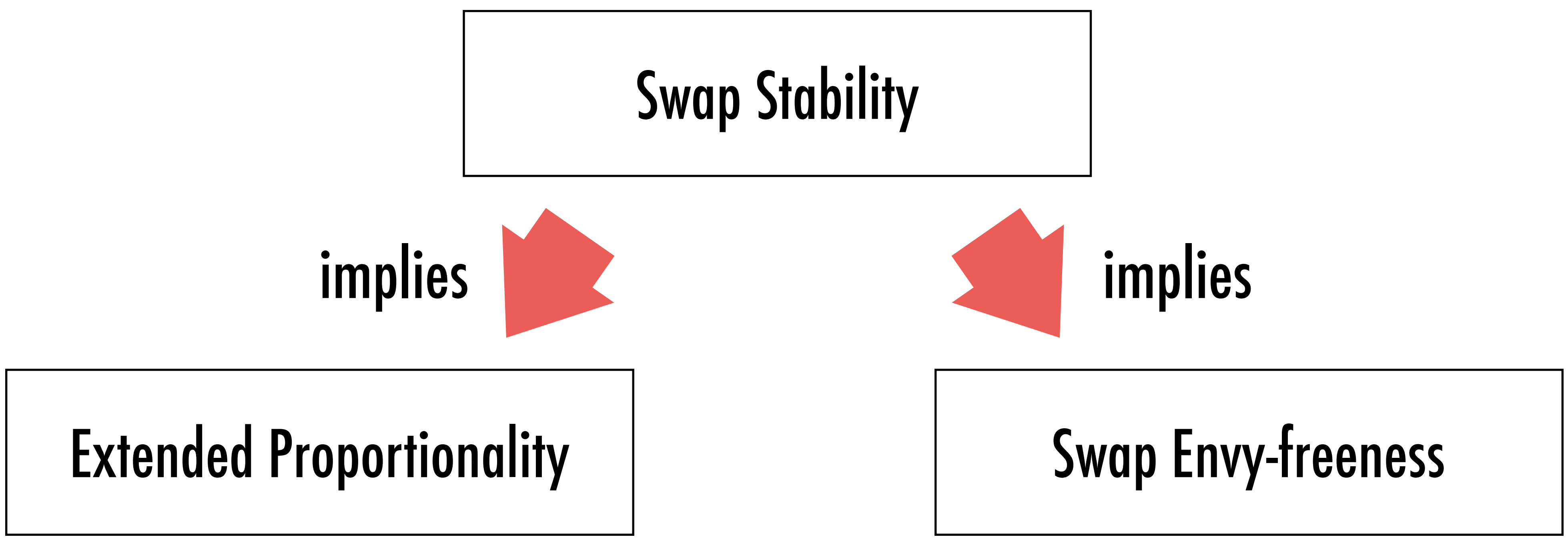
An allocation **A** is swap stable if for every three agents **i**, **j**, and **k** we have

$$V_{j,i}(\mathcal{A}_j) + V_{k,i}(\mathcal{A}_k) \geq V_{j,i}(\mathcal{A}_k) + V_{k,i}(\mathcal{A}_j)$$

Relationship between criteria



Relationship between criteria



But is extended proportionality the best extension of proportionality?

Average Share

Ghodsi et al. (2018)

Consider the average utility agent **i** gains by allocating item **b** to each agent,

$$\bar{V}_i(\{b\}) = \frac{1}{n} \sum_{j \in \mathcal{N}} V_{j,i}(\{b\})$$

Average Share

Ghodsi et al. (2018)

Average Share of agent **i** equals the sum of these average values for all items,

$$\bar{V}_i = \sum_{b \in \mathcal{M}} \sum_{j \in \mathcal{N}} V_{j,i}(\{b\})$$

Average Share

Ghodsi et al. (2018)

An allocation **A** is average

if for each agent we have

$$U_i(\mathcal{A}) \geq \bar{V}_i$$

Average Share vs Extended Proportionality

It is easy to observe that in **network externalities** model, we have the following:

$$\hat{V}_i / n = V_i(\mathcal{M}) \cdot (\max_j w_{j,i}) / n$$

$$\bar{V}_i = V_i(\mathcal{M}) \cdot (\sum_j w_{j,i}) / n$$

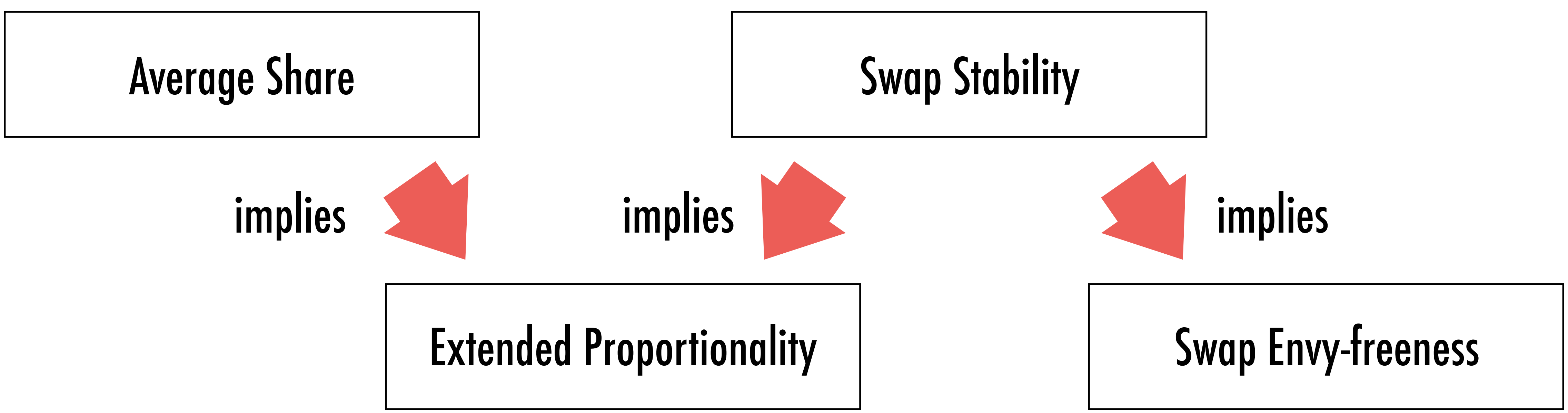
Average Share vs Extended Proportionality

- **Average Share** is more sensitive to externalities in comparison to **Extended Proportionality**.

$$\hat{V}_i / n = V_i(\mathcal{M}) \cdot (\max_j w_{j,i}) / n$$

$$\bar{V}_i = V_i(\mathcal{M}) \cdot (\sum_j w_{j,i}) / n$$

Relationship between criteria



Extended Maximin Share

Ghodsi et al. (2018)

We can utilize the notion of **cut and choose** to find a suitable fairness criterion to capture externalities in fair division of indivisible items.

Extended Maximin Share

Ghodsi et al. (2018)

Cut and choose is consisted of two parts:

1. Division
2. Allocation

Extended Maximin Share

Ghodsi et al. (2018)

1. Division:

   Similar to **Maximin share**, we ask agent **i** to divide items into **n** bundles in a balanced way.

2. Allocation

Note that the valuations is from the point of view of agent i.

Extended Maximin Share

Ghodsi et al. (2018)

1. Division
2. Allocation:

   An **adversary** allocates the bundles to agents in a way that the utility of agent **i** minimizes.

Extended Maximin Share

Ghodsi et al. (2018)

1. Division
2. Allocation:

   An **adversary** allocates the bundles to agents in a way that the utility of agent **i** minimizes.

➡ We call this minimized utility $\mathbf{EMMS_i}$.

Extended Maximin Share

Ghodsi et al. (2018)

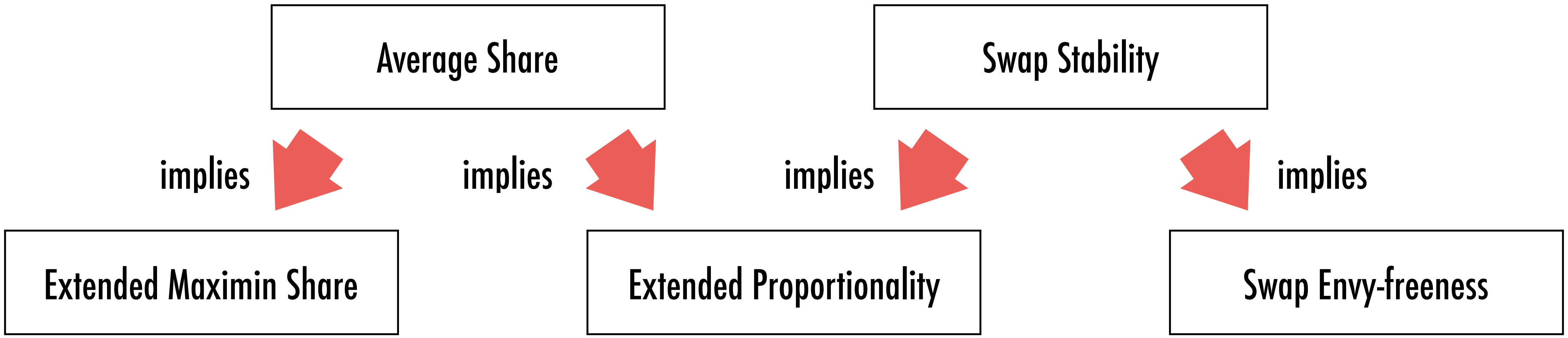
An allocation **A** guarantees Extended Maximin Share, if for each agent we have

$$U_i(\mathcal{A}) \geq \mathrm{EMMS}_i = \max_{P \in \Pi} U_i(\mathcal{W}_i(P))$$

$$\mathcal{W}_i(P) = \arg\min_{\mathcal{A} \in \Omega_P} U_i(\mathcal{A})$$

adversary

Relationship between criteria



Computation Aspects of EMMS

in Network Externalities model

Computing EMMS

We can observe that computing **EMMS** is equivalent to the following problem:

Given a set of items M and a sorted vector of weights w in decreasing order, what is the maximum value of this function if agent i partition M into n bundles where vector x is the sorted values of the bundles in increasing order.

$$w \cdot x = \sum_{i=1}^{n} w_i \cdot x_i$$

Computing EMMS

Given a set of items M and a sorted vector of weights w in decreasing order, what is the maximum value of this function if agent i partition M into n bundles where vector x is the sorted values of the bundles in increasing order.

$$w \cdot x = \sum_{i=1}^{n} w_i \cdot x_i$$

This is the utility agent **i** gains if an **adversary** allocates the bundles.

Computing EMMS

The most common partitioning schemes are the special cases of this problem:

1. Maximin partition $w_1 = 1, w_2 = 0, \ldots, w_n = 0$
2. Minimax partition $w_1 = \frac{1}{n-1}, \ldots, w_{n-1} = \frac{1}{n-1}, w_n = 0$
3. Leximin partition $w_1 = 1 - \epsilon, w_2 = \epsilon - \epsilon^2, \ldots, w_n = \epsilon^{n-1} - \epsilon^n$

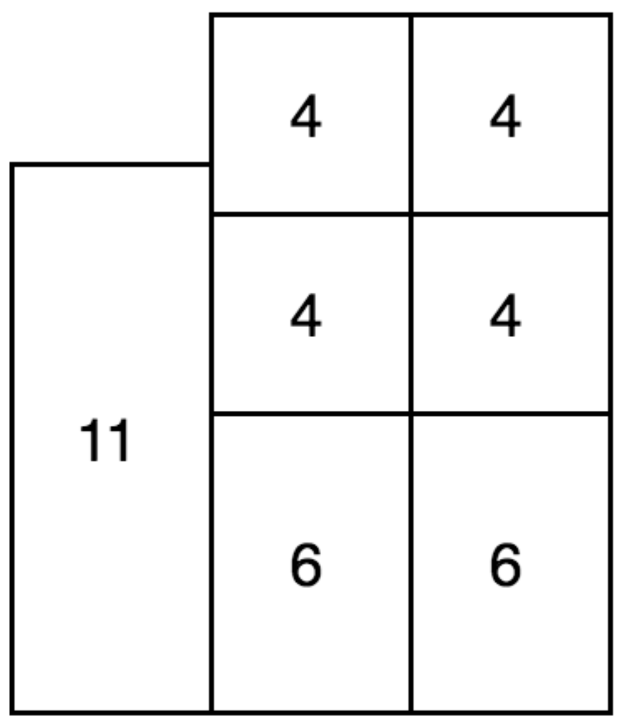
Computing EMMS

The most common partitioning schemes are
the special cases of this problem:

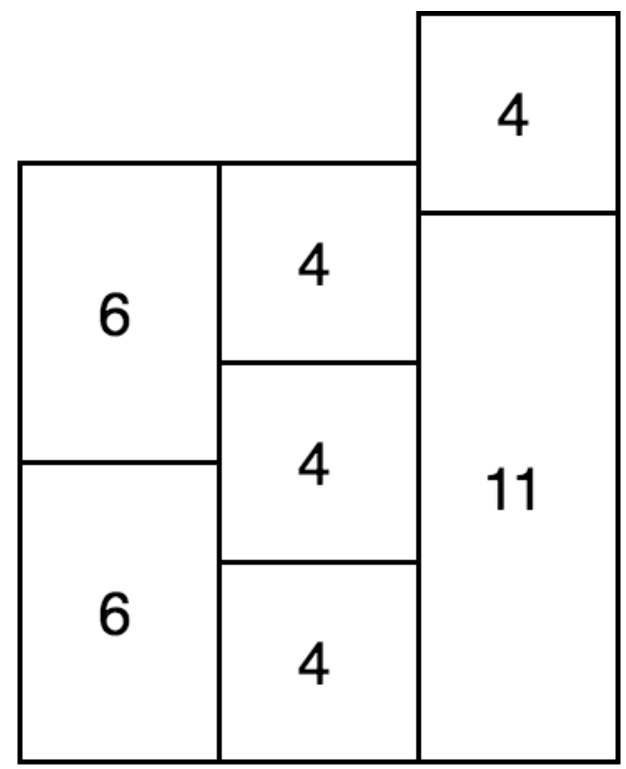
It is **NP-hard** to compute the value of **EMMS**.

Computing EMMS

minimax



>



maximin

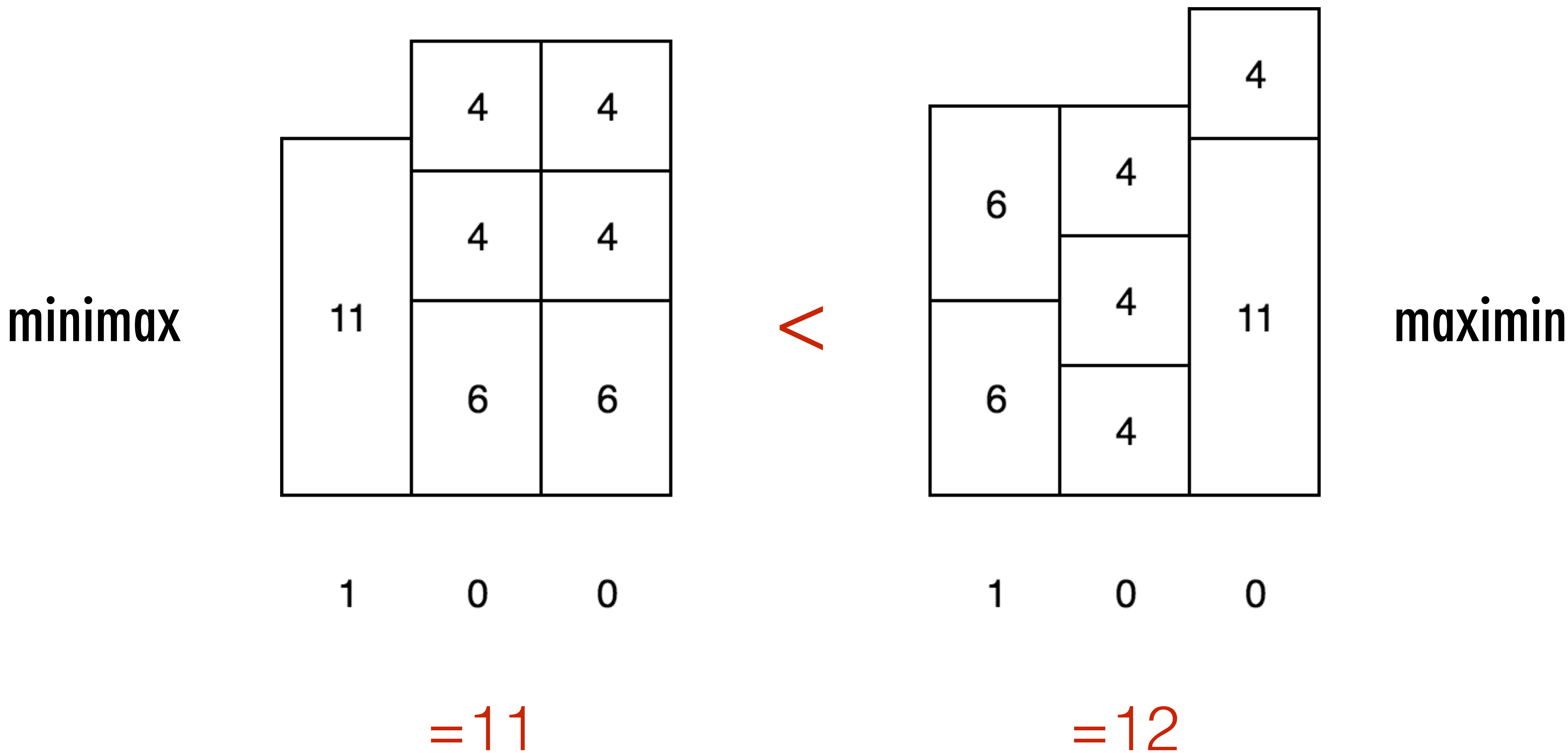
0.5 0.5 0

=12.5

0.5 0.5 0

=12

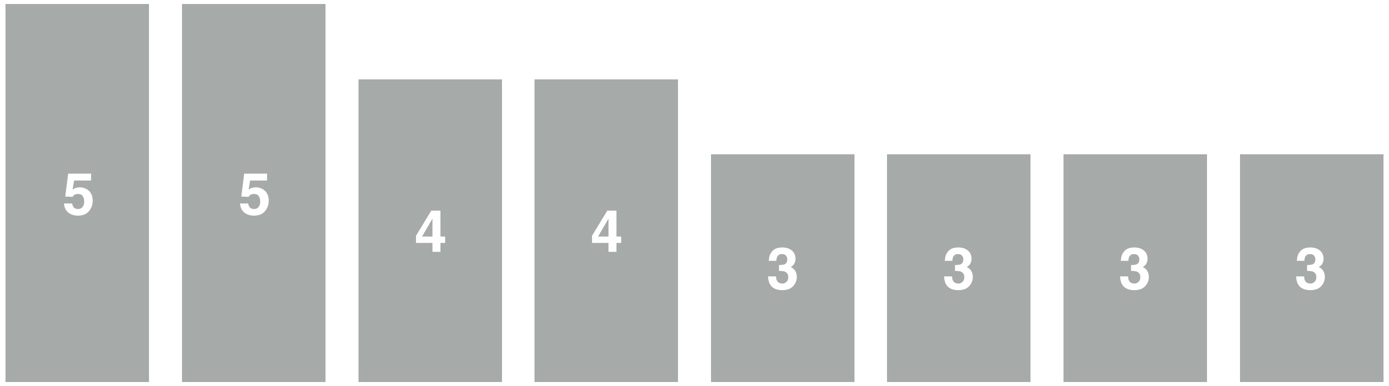
Computing EMMS



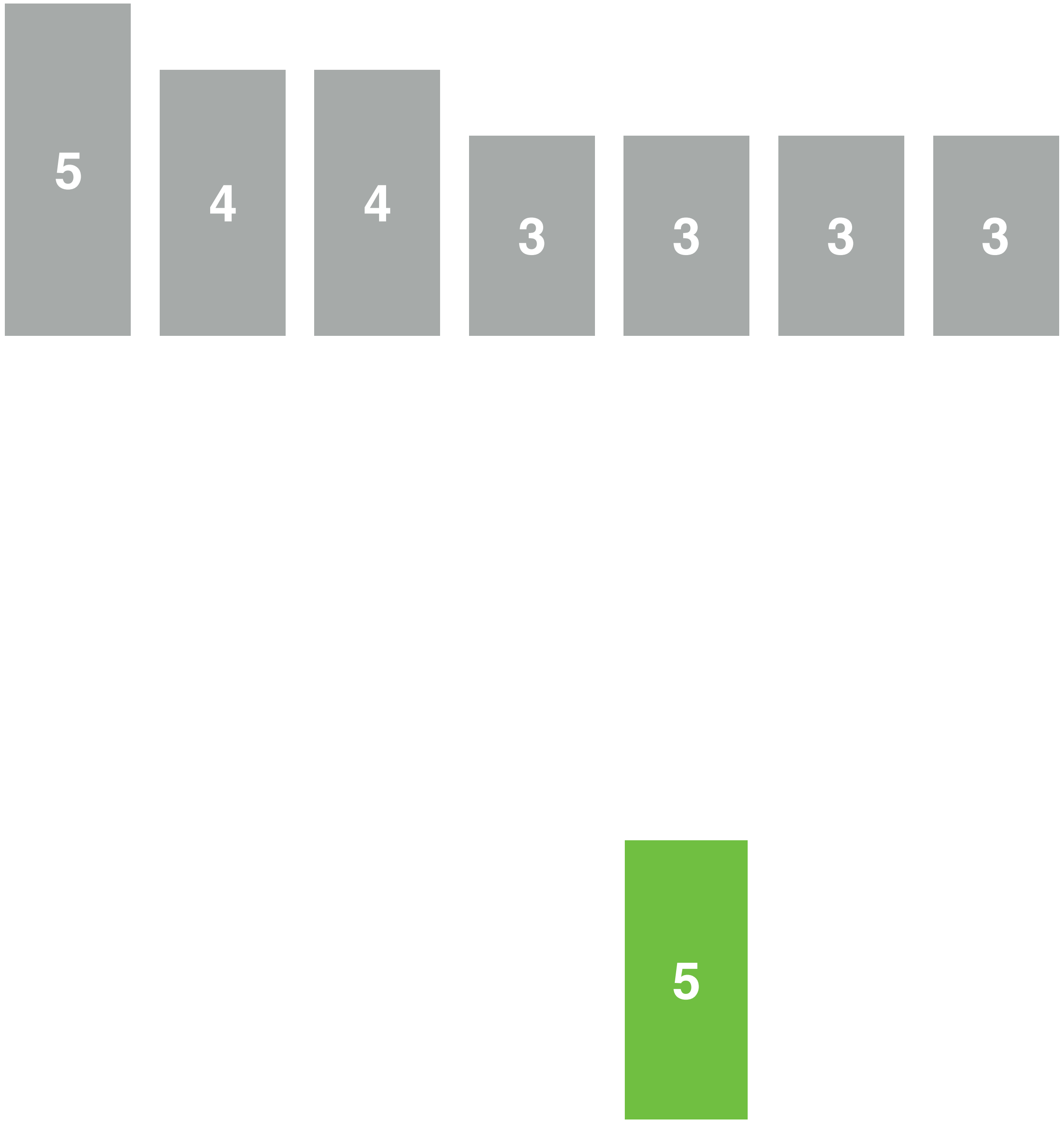
Greedy Approach

A simple greedy algorithm would achieve a 1/2-approximation of the optimum answer.

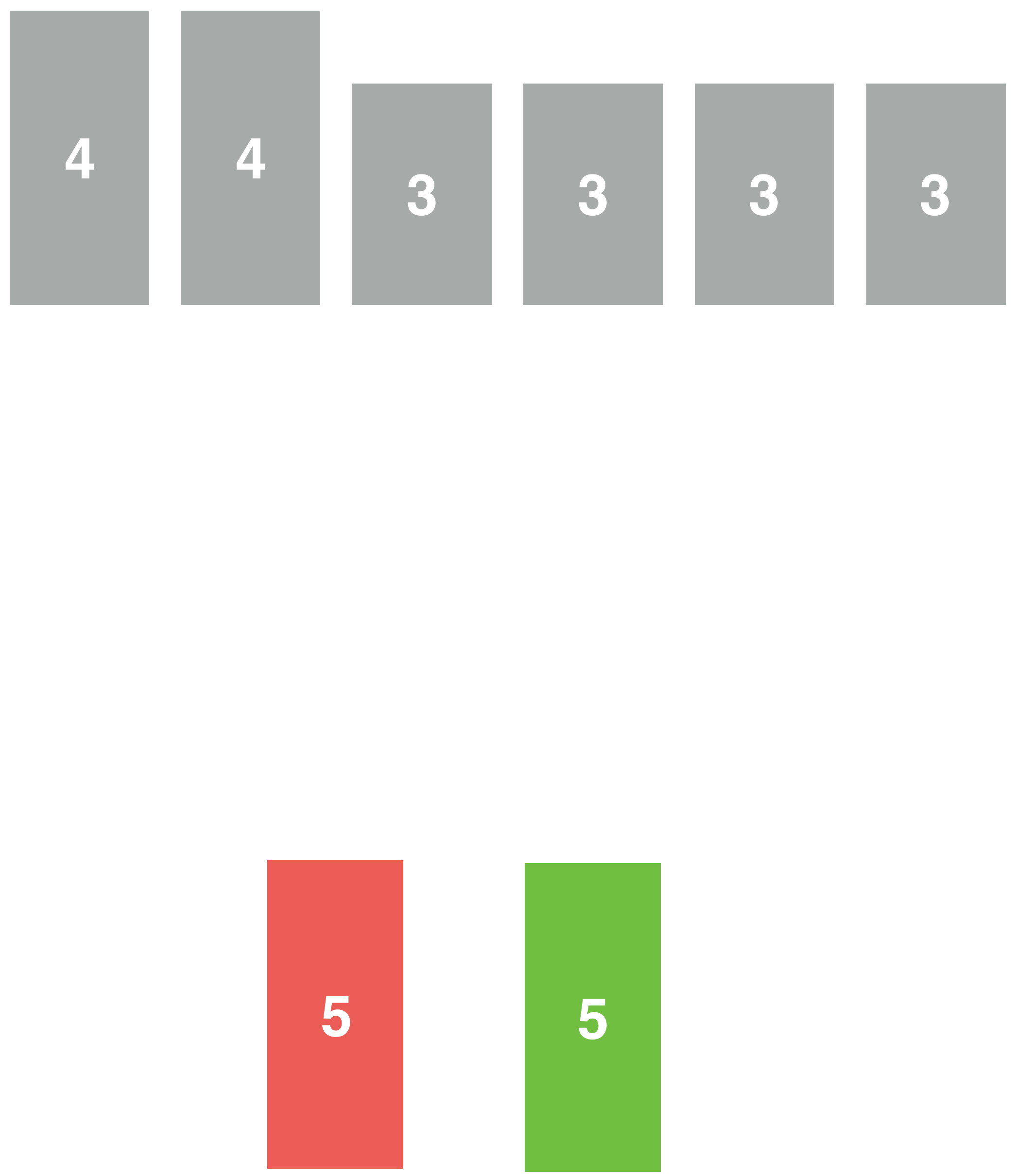
LPT Algorithm



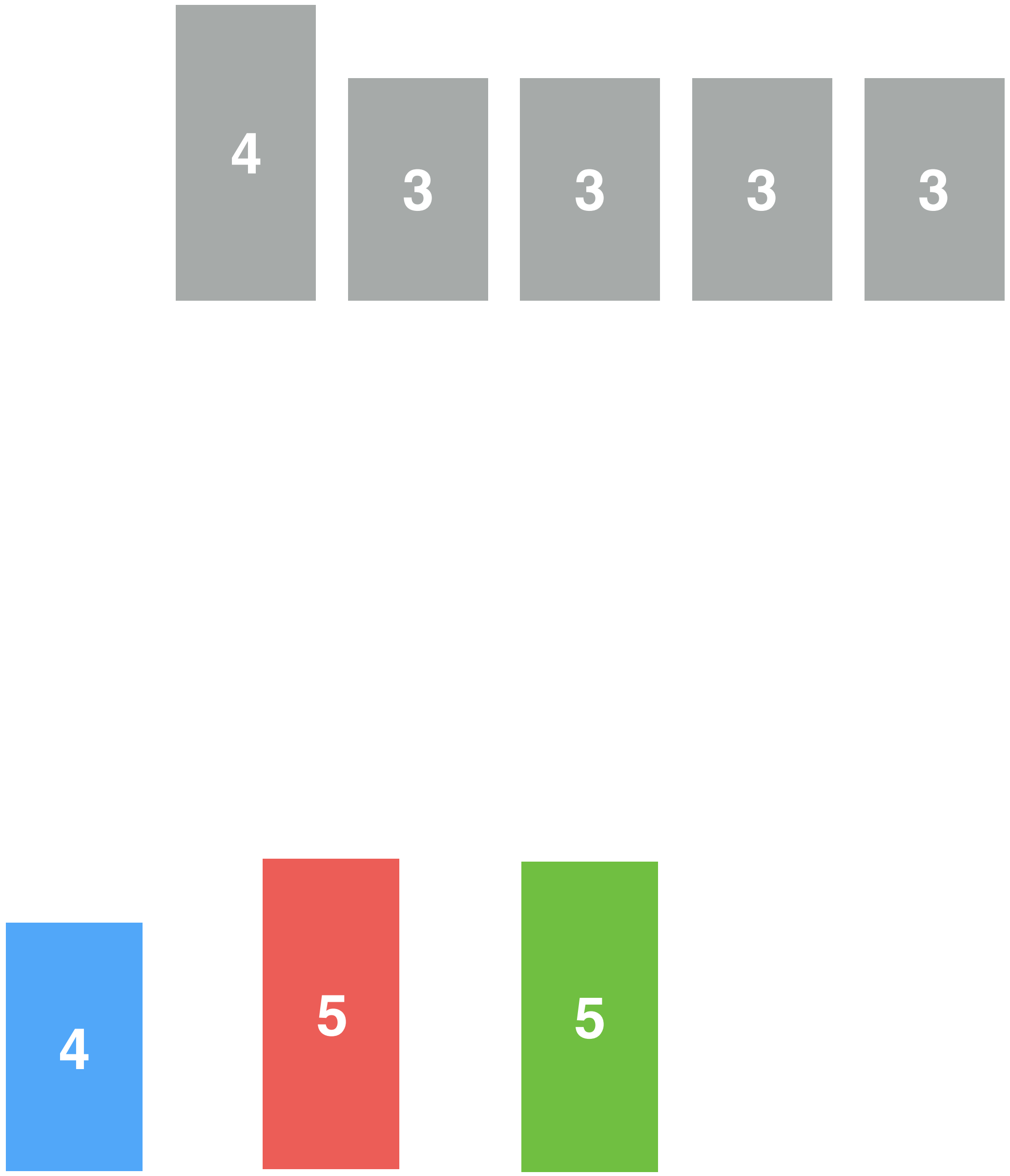
LPT Algorithm



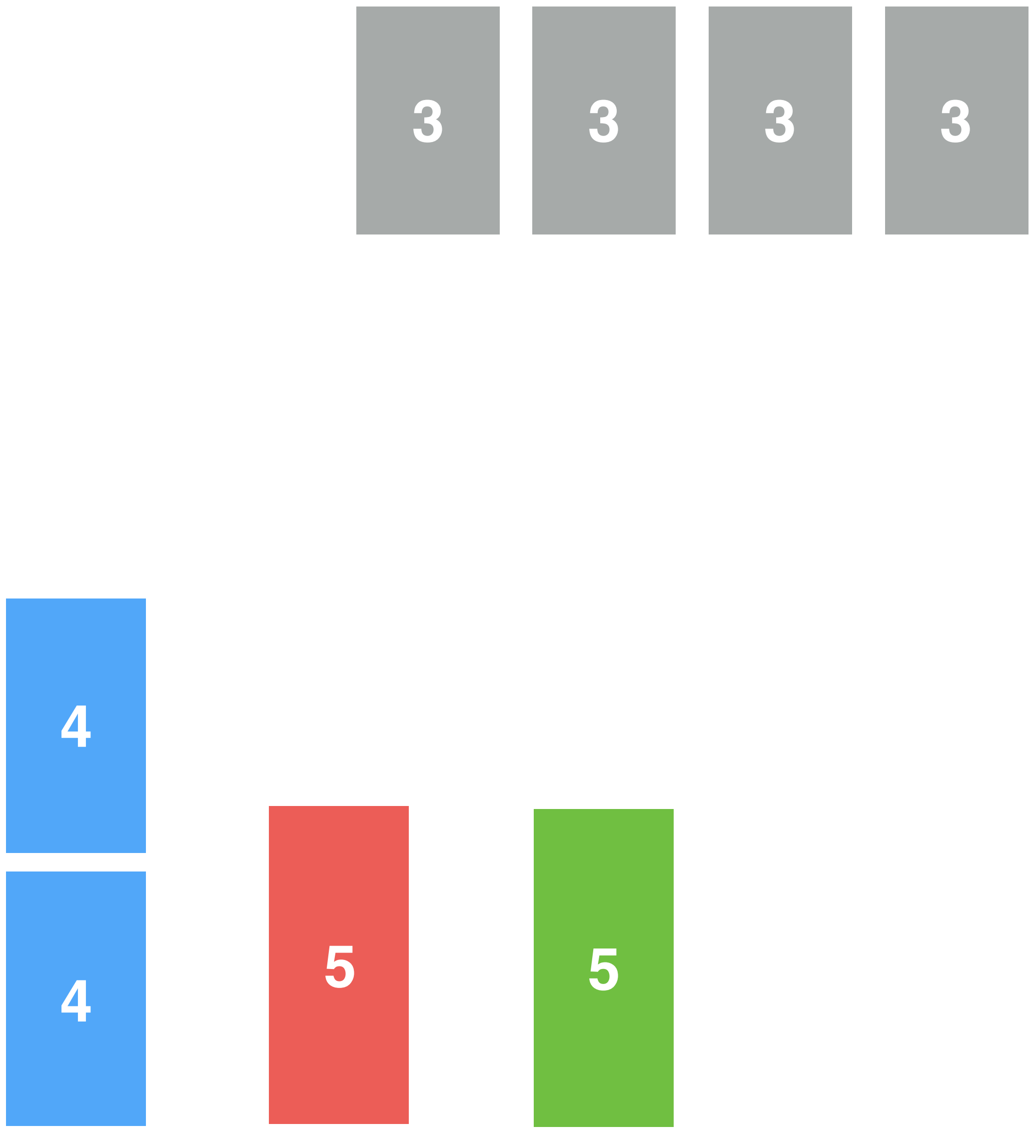
LPT Algorithm



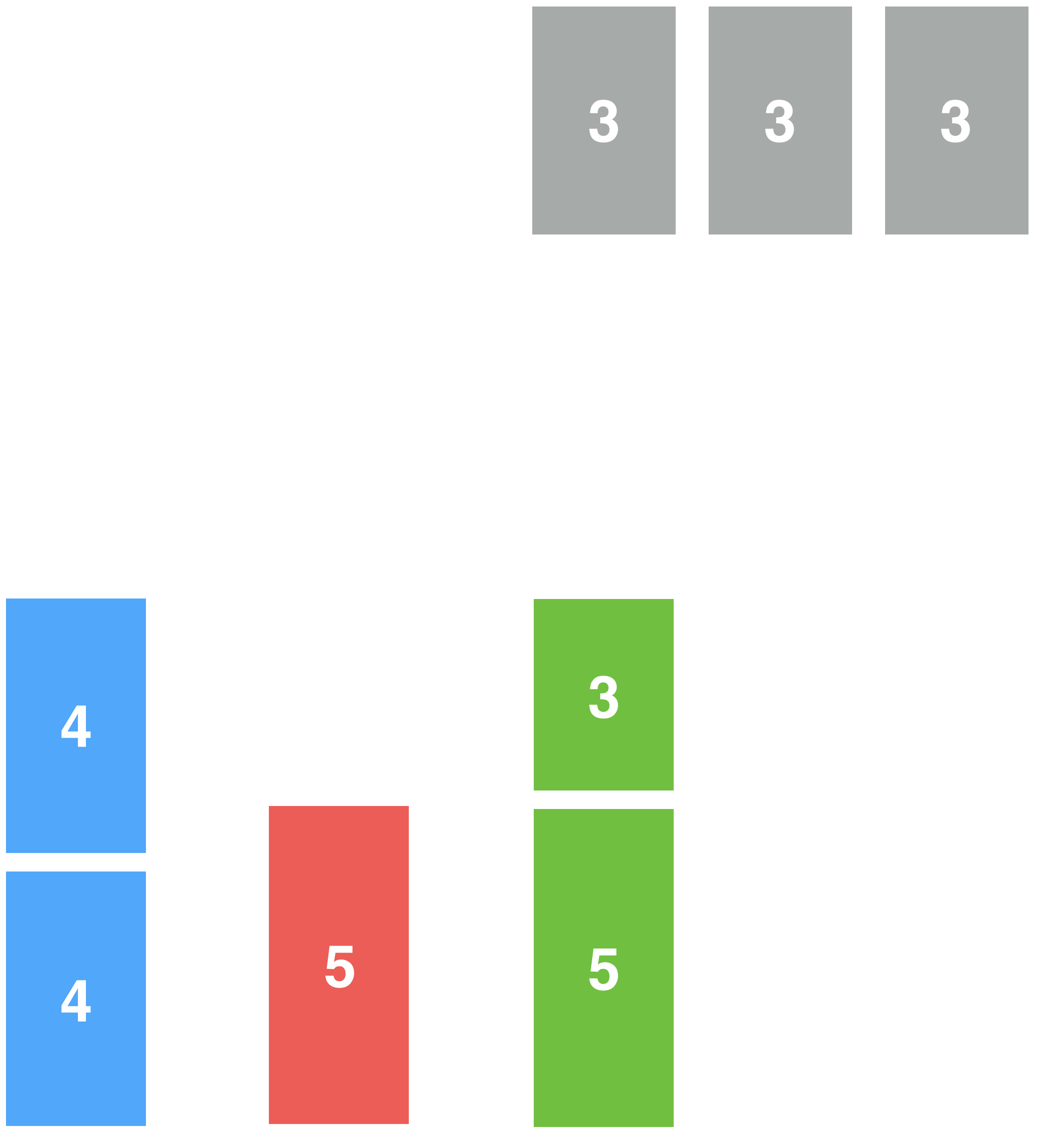
LPT Algorithm



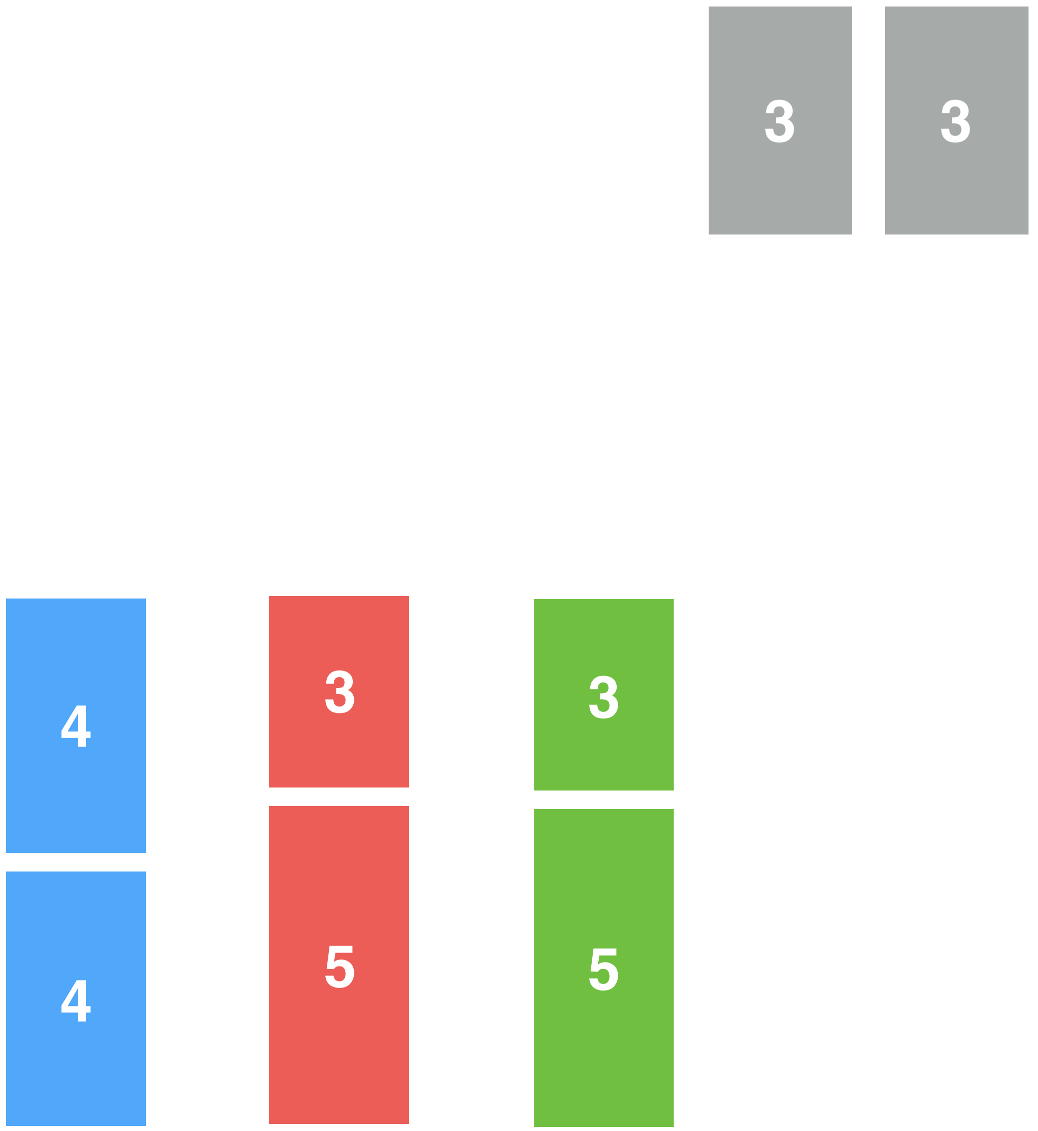
LPT Algorithm



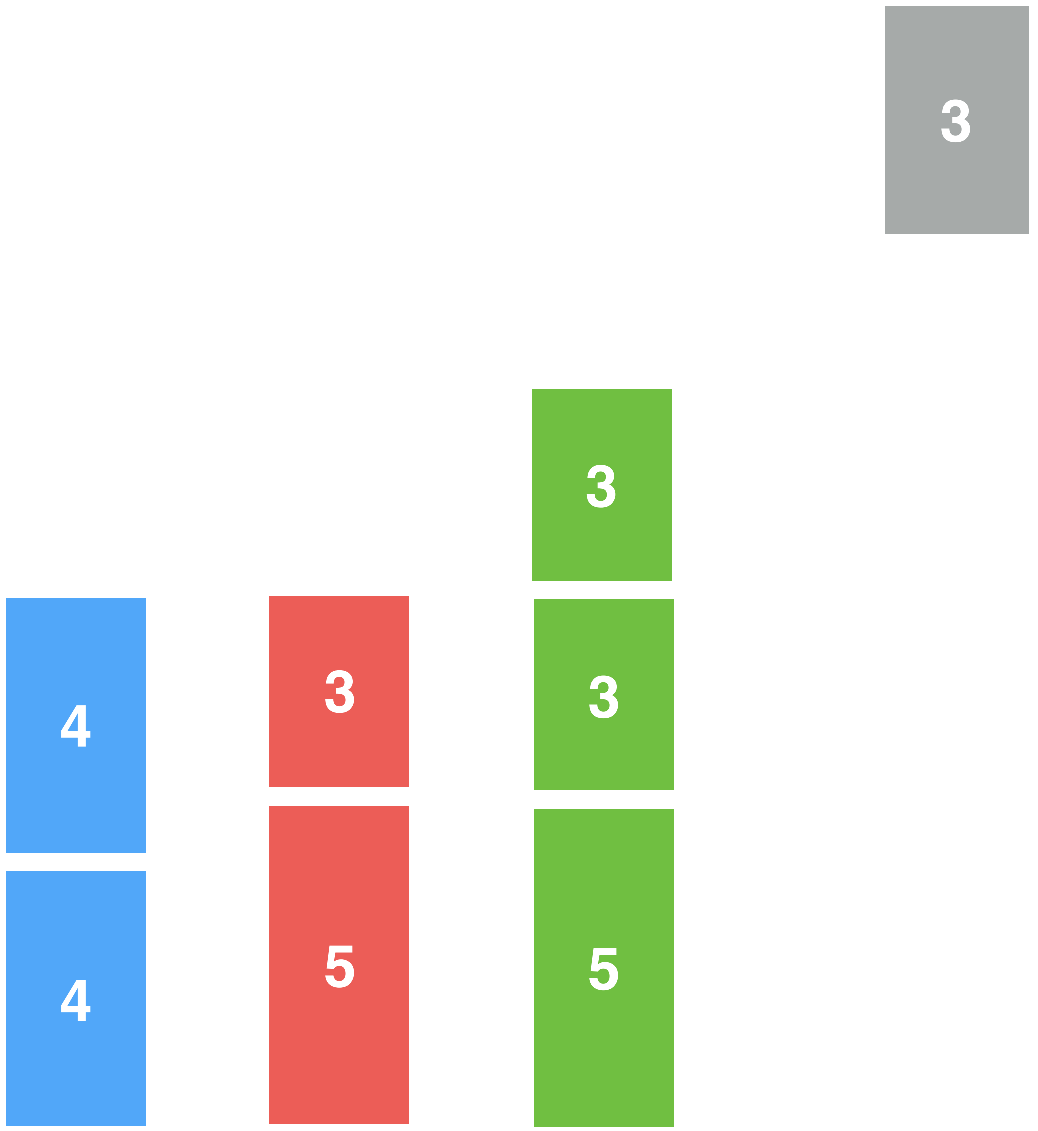
LPT Algorithm



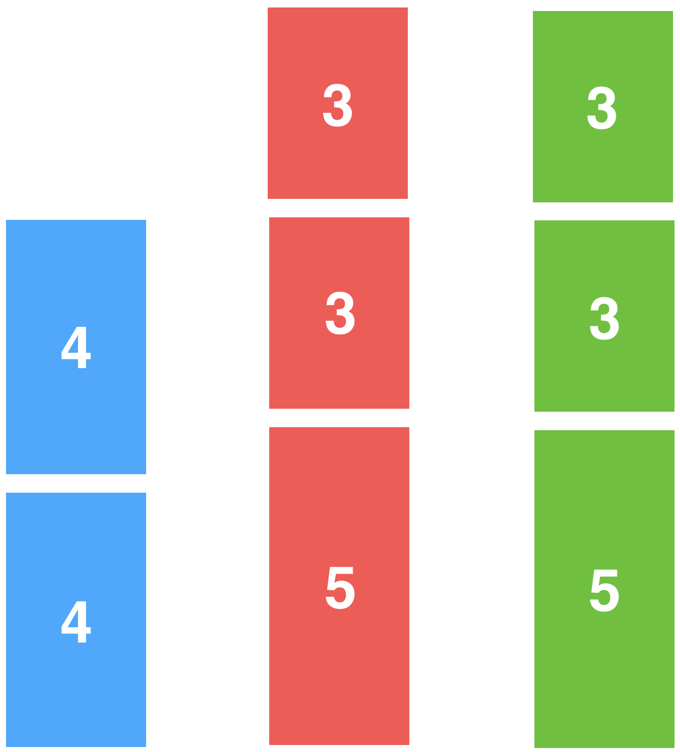
LPT Algorithm



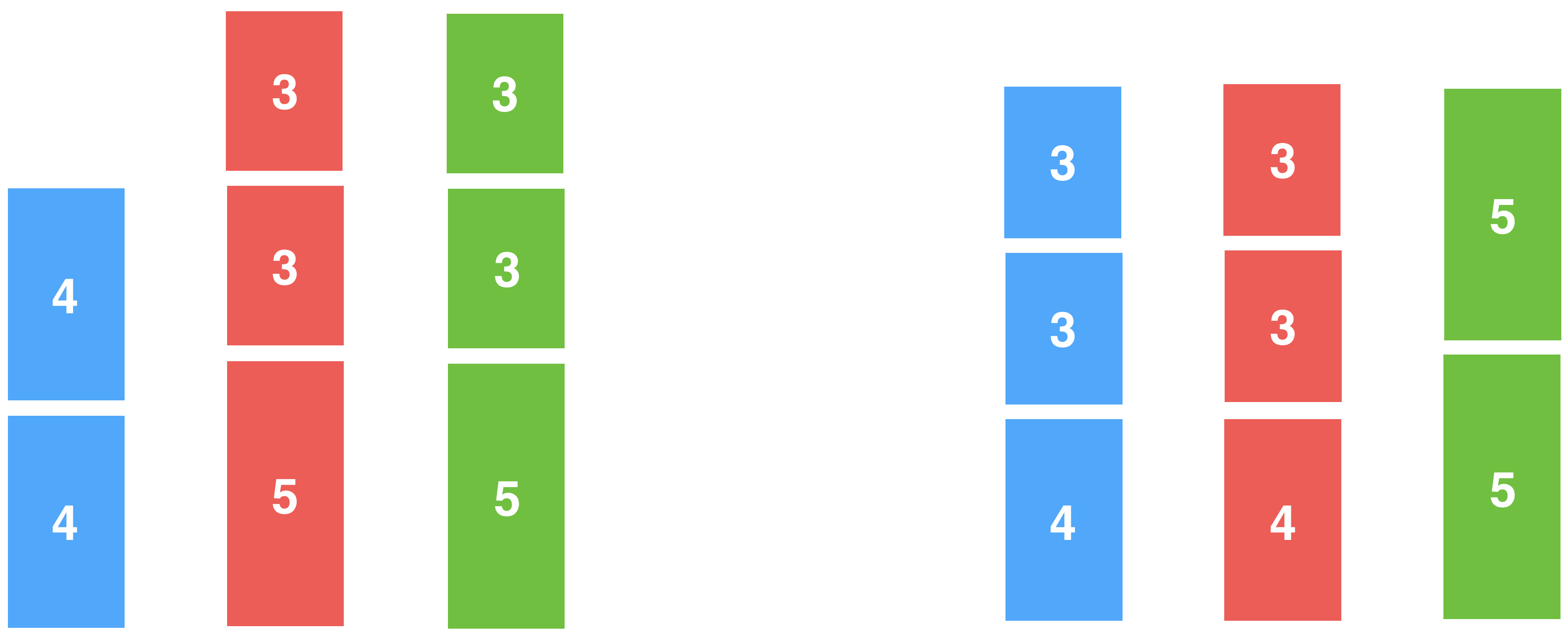
LPT Algorithm



LPT Algorithm



LPT Algorithm



LPT partition

Optimal partition

Computing EMMS

Ghodsi et al. (2018)

Theorem 4.3. The **LPT** algorithms provides a partition which approximates **EMMS** by a factor of 1/2.

Regardless of the weights

Fair Allocation

in Network Externalities model

Self Reliance

We say an agent **i** is **β-self-relient** if

$$w_{i,i} \geq \beta$$

Fair allocation

Ghodsi et al. (2018)

Theorem 5.2. If all the agents are **β-self-relient**, then there exists an allocation that guarantees **β/2EMMS**.

Main result

Fair allocation

Ghodsi et al. (2018)

Corollary 5.6. If all the agents are **β-self-relient**, then we can find an allocation that guarantees **β/4EMMS**.

The algorithm depends on the structure of the **optimal** partition for each agent which we cannot find, but we can use **LPT** partition instead.

Thank you!