University of Maryland
CMSC652 — Complexity Theory
Professor Jonathan Katz

Problem Set 1

Due at the *beginning* of class on Sept. 8

These problems are not meant to be particularly difficult; they are meant more to increase your comfortability with the basic concepts discussed in the first lecture.

1. In class we gave two definitions of $\mathcal{NP}$:

$$\begin{aligned} (1)\ \mathcal{NP} &= \cup_{k \geq 0} \text{NTIME}(n^k) \\ (2)\ \mathcal{NP} &= \{L \mid \exists \text{ poly-time relation } R \text{ s.t. } L = L_R\}. \end{aligned}$$

   Prove that these definitions are equivalent.

2. Prove that if L is $\mathcal{NP}$-complete and $L \in \mathcal{P}$ then $\mathcal{P} = \mathcal{NP}$.

3. A notion of $\mathcal{P}$-completeness can be defined in a manner similar to $\mathcal{NP}$-completeness. Consider the following definition (*be warned*: this definition is *not* the standard one):

   > A language L is $\mathcal{P}$-complete if (1) $L \in \mathcal{P}$ and (2) for any $L' \in \mathcal{P}$, there is a Karp-reduction from L' to L.

   Show that the language

$$L = \left\{ \langle M, x, 1^t \rangle \mid \begin{array}{c} M \text{ is a deterministic T.M.} \\ \text{which accepts } x \text{ within } t \text{ steps} \end{array} \right\}$$

   is $\mathcal{P}$-complete.

4. Show that Karp reductions are transitive: namely, that if $L_1 \leq_m^P L_2$ and $L_2 \leq_m^P L_3$ then $L_1 \leq_m^P L_3$.

5. (optional) We may re-phrase the definition of $\mathcal{NP}$-completeness we gave in class as follows:

   > L is $\mathcal{NP}$-complete if (1) $L \in \mathcal{NP}$ and (2) for all $L' \in \mathcal{NP}$ there exists a function f and a polynomial poly such that (2a) $f(x)$ is computable in time $\mathsf{poly}(|x|)$ and (2b) $x \in L'$ iff $f(x) \in L$.

   Consider the following variant in which the order of quantifiers is changed:

   > L is super-$\mathcal{NP}$-complete if (1) $L \in \mathcal{NP}$ and (2) there exists a polynomial poly such that for all $L' \in \mathcal{NP}$ there exists a function f such that (2a) $f(x)$ is computable in time $\mathsf{poly}(|x|)$ and (2b) $x \in L'$ iff $f(x) \in L$.

   Prove that super-$\mathcal{NP}$-complete languages do not exist. (*Hint*: use the fact that an analogue of the time hierarchy theorem holds for non-determinisic time classes.)