NP-Completeness: The Equivalence of the Decision and Optimization Independent Set Problems

Coloring Problem: Optimization Version

Let $G = (V, E)$ be an undirected graph. An independent set of G is a subset of the vertices such that so that no two vertices in the subset are neighbors (in G).

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Time $O(n^r) + O(n) = O(n^r)$. □

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Given optimization problem for undirected graph $G = (V, E)$:

Some magic.

□

Algorithm

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```
/* Find optimal number of vertices */
k ← n
while not IS_Decision(G,k) do
    k ← k-1
end while
```

Algorithm continued

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```
        /* Find the vertices */
Independent_Set ← ∅
for i = 1 to n do
    G' ← G
    delete vertex i and its neighbors from G'
    if IS_Decision(G',k-1) then
        Independent_Set ← Independent_Set ∪ {i}
        k ← k-1
        G ← G'
    end if
end for
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Analysis

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$$T(n) \;=\; O(nn^s)$$

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\begin{aligned}
T(n) &= O(nn^s) \\
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Time $O(n^r) + O(n) \; = \; O(n^r)$. □

The other way around.

Idea for algorithm

Assume that given $G = (V, E)$,
`partial_colorable(G,k,p)` decides if the partial coloring p of G can be extended to a k-coloring of G in time $O(n^s)$.

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p ← empty coloring
k ← 1
while not partial_colorable(G,k,p) do
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`partial_colorable is not the decision version!`

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Another great Rumsfeld quote:

> *There are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns ...*

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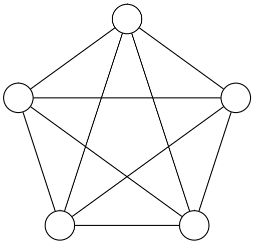

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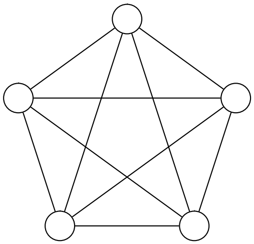

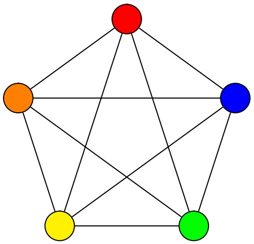

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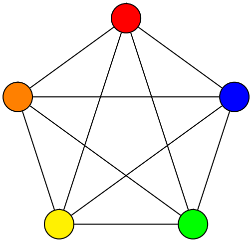

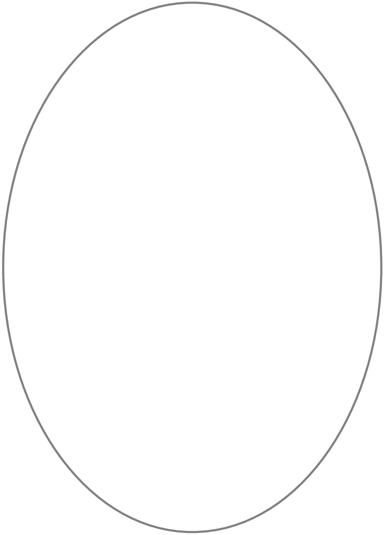

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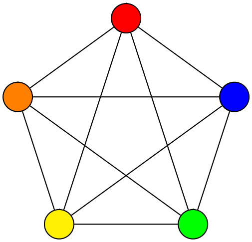

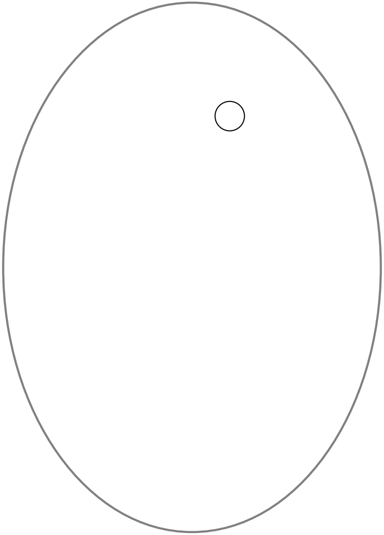

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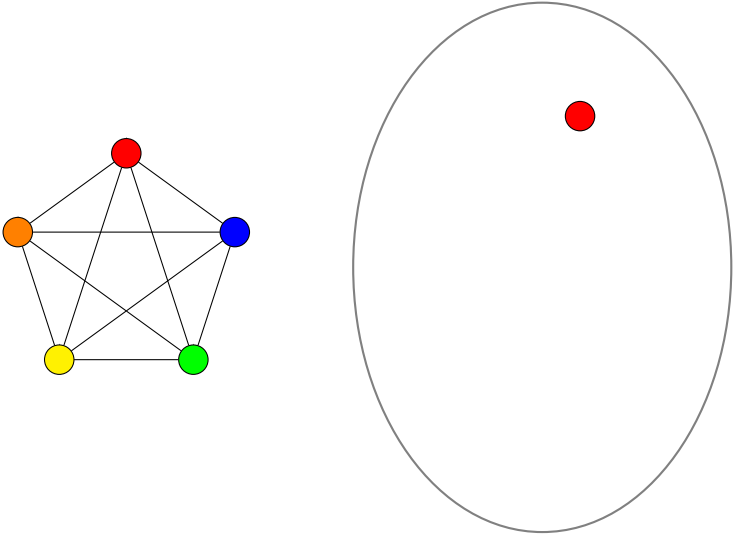

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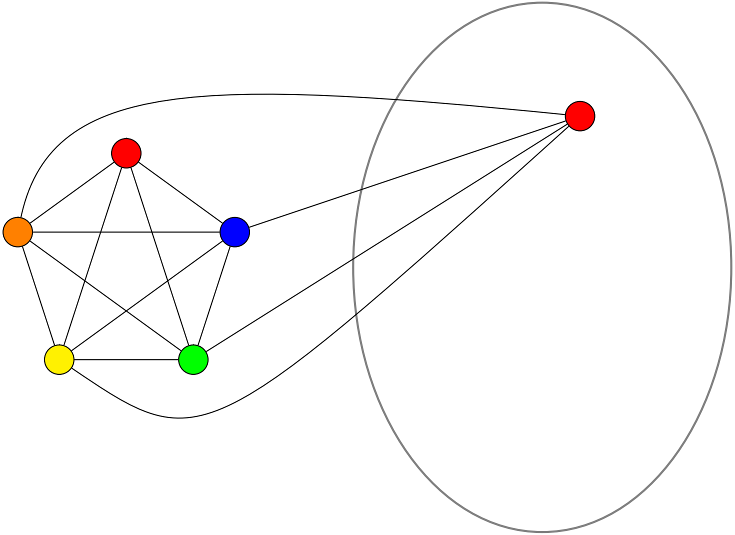

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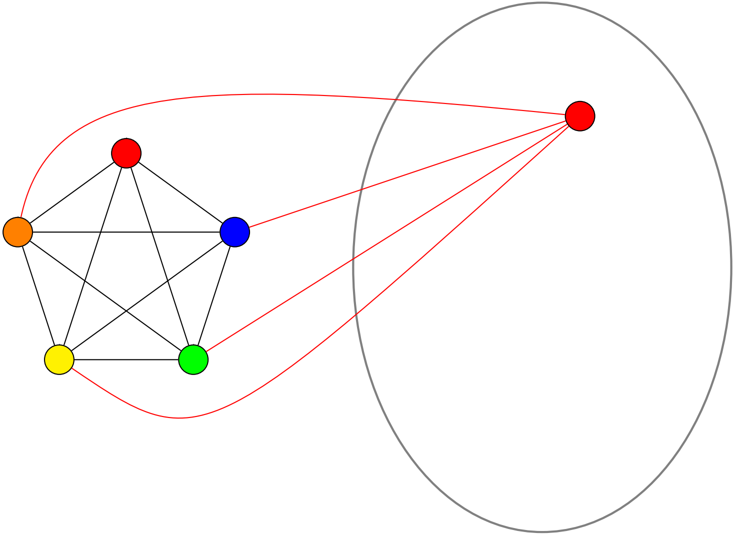

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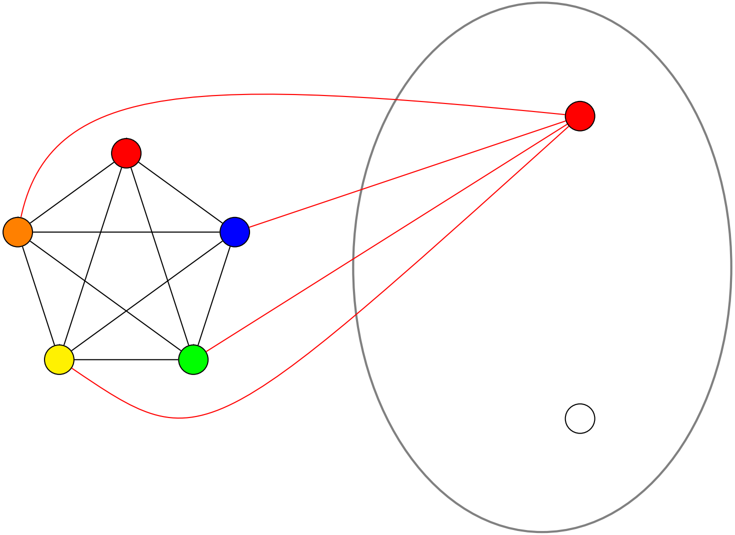

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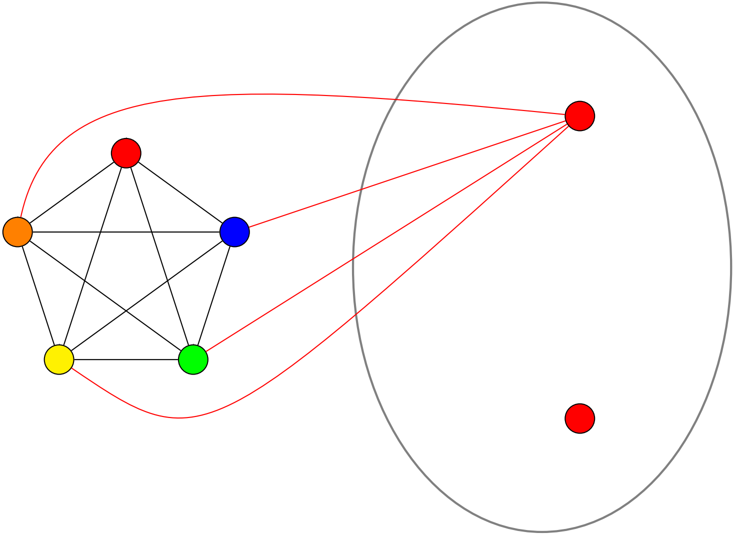

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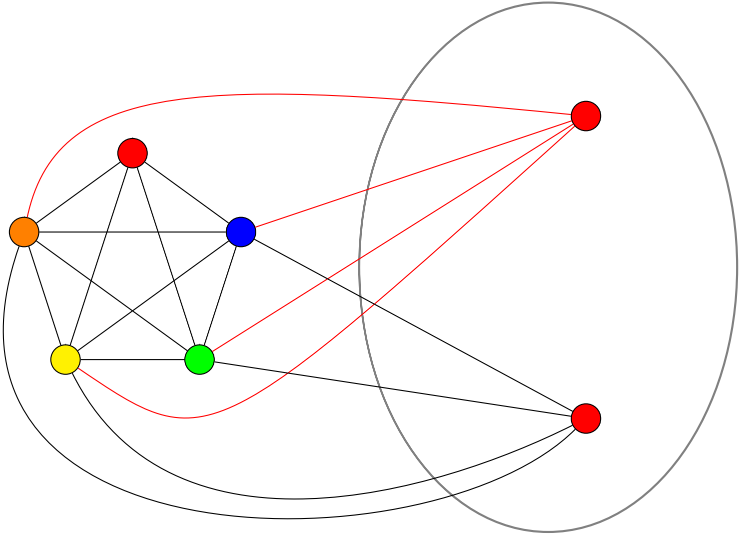

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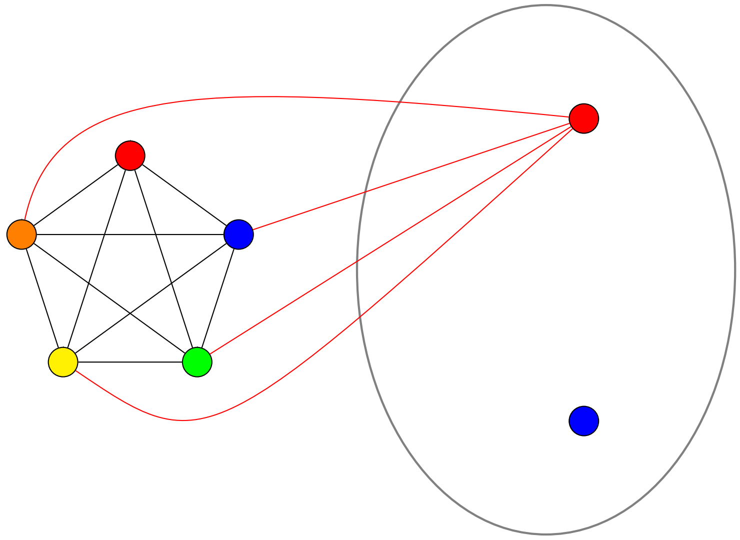

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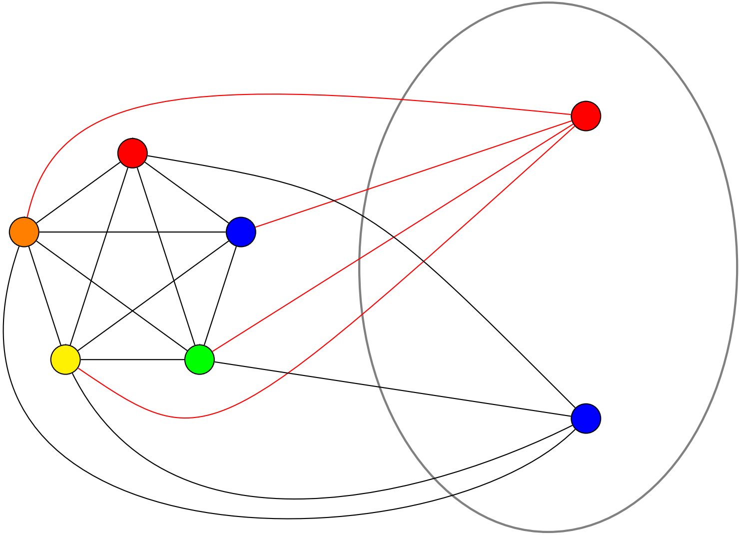

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$O(nk(k + (n + k)^s))$

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$$T(n) \; = \; O(kn^s) \; + \; O(k^2) \; + \; O(nk(k+(n+k)^s))$$

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&= O(n^{s+1}) + O(n^2) + O(n^3 + n^2(2n)^s)) \\
&= O(n^{s+1}) + O(n^2) + O(n^3 + n^2 2^s n^s) \\
&= O(n^{s+1}) + O(n^2) + O(n^3 + n^{s+2}) \\
&= O(n^{s+2})
\end{aligned}
$$