

Improved Approximation Bounds for Planar Point Pattern Matching

Minkyong Cho · David M. Mount

Received: 6 January 2006 / Accepted: 11 December 2006 / Published online: 13 October 2007
© Springer Science+Business Media, LLC 2007

Abstract We analyze the performance of simple algorithms for matching two planar point sets under rigid transformations so as to minimize the directed Hausdorff distance between the sets. This is a well studied problem in computational geometry. Goodrich, Mitchell, and Orletsky presented a very simple approximation algorithm for this problem, which computes transformations based on aligning pairs of points. They showed that their algorithm achieves an approximation ratio of 4. We introduce a modification to their algorithm, which is based on aligning midpoints rather than endpoints. This modification has the same simplicity and running time as theirs, and we show that it achieves a better approximation ratio of roughly 3.14. We also analyze the approximation ratio in terms of a instance-specific parameter that is based on the ratio between diameter of the pattern set to the optimum Hausdorff distance. We show that as this ratio increases (as is common in practical applications) the approximation ratio approaches 3 in the limit. We also investigate the performance of the algorithm by Goodrich et al. as a function of this ratio, and present nearly matching lower bounds on the approximation ratios of both algorithms.

Keywords Point pattern matching · Approximation algorithms · Hausdorff distance

This work was supported by the National Science Foundation under grants CCR-0098151 and CCF-0635099.

M. Cho (✉)

Department of Computer Science, University of Maryland, College Park, MD 20742, USA
e-mail: minkcho@cs.umd.edu

D.M. Mount

Department of Computer Science and Institute for Advanced Computer Studies,
University of Maryland, College Park, MD 20742, USA
e-mail: mount@cs.umd.edu

1 Introduction

Geometric point pattern matching problem is a fundamental computational problem and has numerous applications in areas such as computer vision [19], image and video compression [4], model-based object recognition [15], and computational chemistry [10]. Two point sets are given from some space, a *pattern set* P and *background set* Q . Let $m = |P|$ and $n = |Q|$. The goal is to compute the transformation E from some geometric group acting on this space that minimizes some distance measure from the transformed pattern set $E(P)$ to the background set Q . Throughout, we will consider point sets in the Euclidean plane under rigid transformations (translation and rotation). Distances between two point sets P and Q will be measured by the *directed Hausdorff distance*, denoted $h(P, Q)$, which is defined to be

$$h(P, Q) = \max_{p \in P} \min_{q \in Q} \|pq\|,$$

where $\|pq\|$ denotes the Euclidean distance between points p and q .

The directional Hausdorff distance is natural in applications where the pattern P is expected to match some subset of the background Q . In applications where it is desired that every point of P matches some point of Q and vice versa, a bidirectional similarity measure may be more appropriate such as the *bidirectional Hausdorff distance*, which is defined to be $\max(h(P, Q), h(Q, P))$. When matching with bidirectional similarity measures it may be possible to identify a global reference point (such as the centroid of each set) about which to anchor the alignment. This is not possible, however, under the directed Hausdorff distance. Throughout, unless otherwise specified, we use the term *Hausdorff distance* to denote the directed Hausdorff distance.

The problem of point pattern matching has been widely studied. We will focus on methods from the field of computational geometry. Excellent surveys can be found in [2, 21]. There are many variants of this problem, depending on the nature of the inputs, the allowable group of aligning transformations and the distance function employed. Some formulations differ from ours either in that they require exact matches [8, 20] or they involve bidirectional similarity measures [1, 3, 9, 14, 18].

A number of algorithms have been proposed for computing optimal pattern matching under the Hausdorff distance [6, 16]. The computational complexity can be quite high, however. For example, the best-known algorithm for determining the translation and rotation that minimize the directed Hausdorff distance between two planar point sets P and Q of sizes m and n , respectively, runs in $O(m^3 n^2 \log^2 mn)$ time [6]. This complexity may be unacceptably high for applications involving hundreds or thousands of points. For this reason approximation algorithms have been considered. In this context the *approximation ratio* of a pattern matching algorithm is the maximum ratio over all input instances between the Hausdorff distance produced by the algorithm and the optimum Hausdorff distance.

A simple and natural approach is to consider transformations induced by aligning a small subset of points from one set to the other. This is arguably the simplest and most easily implemented algorithm for approximate pattern matching, and it is the basis of some of the most popular methods in computer vision such as RANSAC [11]. Goodrich et al. [13] were the first to prove an upper bound on the approximation ratio

of such a simple alignment-based algorithm. They considered point pattern matching under a number of different transformation spaces and in different dimensions. For the case of rigid transformations in the plane, their algorithm computes a diametrical pair for P and then computes for every pair of distinct points of Q a rigid transformation that aligns these pairs. (Details are presented in the next section.) It returns the transformation achieving the minimum Hausdorff distance. It runs in $O(n^2 m \log n)$ time. They prove that it returns an aligning transformation whose Hausdorff distance is at most a factor of 4 larger than the optimum Hausdorff distance.

Other authors have considered more precise approximation algorithms for planar point matching. Indyk, Motwani, and Venkatasubramanian [17] presented an ε -approximation algorithm for this problem, that is, one whose Hausdorff distance is most a factor of $(1 + \varepsilon)$ greater than the optimum, where $\varepsilon > 0$ is a user-supplied parameter. Their formulation differs somewhat from ours, since it involves a decision problem that is given an estimate of the optimum Hausdorff distance. Their approach is also based on alignments, but it achieves higher accuracy than that of [13] by generating a number of aligning transformations for each aligning pair. Their algorithm's running time is not purely combinatorial, but depends on the *spread* of the set, denoted Δ , which is defined to be the ratio of the distances between the farthest and closest pairs of points. They showed that, for any fixed $\varepsilon > 0$, a $(1 + \varepsilon)$ -approximation can be computed in time $O(n^{4/3} \Delta^{1/3} \log n)$, where the hidden constant factor depends on ε . Cardoze and Schulman [5] also presented an efficient randomized ε -approximation algorithm, which achieves its efficiency by reduction to a number of 1-dimensional point pattern matching problems, each of which is then reduced to computing a convolution using the fast Fourier transform. Its running time is $O(n^2 \log n + \log^{O(1)} \Delta)$ for any fixed precision parameter and any fixed success probability. Gavrilov et al. [12] present an empirical study of a number of pattern matching algorithms.

The purpose of this paper is not to propose a new algorithm, but rather to tighten the analysis of the approximation factors achievable by alignment-based approaches. Our focus will be on the alignment-based algorithm of Goodrich, Mitchell, and Orletsky [13]. Note that our results can also be used to improve the performance of other algorithms that make use of matches based on point-to-point alignments, such as the ε -approximation algorithm by Indyk, Motwani, and Venkatasubramanian [17]. The algorithm of Goodrich et al. is based on aligning points one by one, and henceforth we refer to this as *serial alignment*. (We will describe this algorithm in detail in Sect. 2.) Given the simplicity of this approach, it is natural to establish good bounds on its performance. This also points toward larger questions such as what are the best approaches to alignment-based matching and what are the inherent limits on the accuracy of alignment-based matching. We show that it is possible to improve on their approximation ratio of 4. Our approach has the same running time as theirs and, like theirs, is very easy to implement. It is based on a minor modification that selects the transformation that best aligns the entire subset of points, which we call *symmetric alignment*. Let A_{ser} and A_{sym} denote the approximation ratios for these respective algorithms.

Rather than just considering the worst-case approximation ratios, we analyze the approximation ratios of these algorithms in a manner that is sensitive to the optimal

Table 1 Summary of results: Bounds on the approximation ratios for symmetric and serial alignment algorithms as a function of the distance ratio ρ , where $c_\infty \approx 3.19$

Algorithm	Approximation ratio		
	Upper bound (all ρ)	Upper bound	Lower bound
Symmetric alignment	$A_{\text{sym}} \leq 3.14$	$A_{\text{sym}}(\rho) \leq 3 + \frac{1}{\sqrt{3}\rho}$	$A_{\text{sym}}(\rho) \geq 3 + \frac{1}{10\rho^2}$
Serial alignment	$A_{\text{ser}} \leq 3.44$	$A_{\text{ser}}(\rho) \leq c_\infty + \frac{9}{4\rho}$	$A_{\text{ser}}(\rho) \geq c_\infty + \frac{1}{27\rho^2}$

Hausdorff distance. For each problem instance P and Q , we define an geometric parameter ρ , called the *distance ratio*, to be half the ratio of the diameter of P to the optimum Hausdorff distance between P and Q . (A formal definition is given in Sect. 3.) We show that as the distance ratio increases, the accuracy of the approximation increases as well. We feel that this analysis is useful because large values of ρ often arise in applications. For example, in document analysis and satellite image analysis, the ratio of the diameter of a typical pattern ranges from tens to hundreds of pixels, while the expected digitization error is on the order of a single pixel. Let $A_{\text{ser}}(\rho)$ and $A_{\text{sym}}(\rho)$ denote the approximation ratios for serial and symmetric alignment, respectively, as a function ρ . Our results are summarized in Table 1. The quantity c_∞ is the approximation ratio $A_{\text{ser}}(\rho)$ in the limit as ρ approaches infinity, which we show later to be roughly 3.19.

Our results imply that the approximation ratio for symmetric alignment is better than that of serial alignment for almost all but very small values of ρ . Further, for typical applications where distance ratio is large, the approximation factor of symmetric alignment is close to 3. As mentioned earlier, our results can be applied to other algorithms that are based on point-to-point alignments. For example, the ε -approximation algorithm of Indyk et al. [17] uses the simple alignment algorithm as a subroutine. The running time of their algorithm has a cubic dependence on the approximation ratio of the alignment algorithm. So, improving the approximation ratio bound by a factor of f results in factor of f^3 reduction in the running time of their algorithm.

The remainder of the paper is organized as follows. In Sect. 2 we present the serial and symmetric alignment algorithms, and we provide an overview of their approximation ratios. In Sects. 3 and 4, we present our upper and lower bounds, respectively, on the approximation ratio for symmetric alignment. In Sects. 5 and 6, we present upper and lower bounds for serial alignment. Finally, in Sect. 7 we summarize and present concluding remarks.

2 The Serial and Symmetric Alignment Algorithms

In this section we present a description of the serial alignment algorithm of Goodrich, Mitchell, and Orletsky [13] and review its approximation ratio. We also described our modification of this algorithm, called symmetric alignment. Recall that we are given two planer point sets, a pattern set $P = \{p_1, \dots, p_m\}$ and a background set $Q = \{q_1, \dots, q_n\}$. We seek a rigid transformation E that minimizes the (directed) Hausdorff distance from $E(P)$ to Q .

The serial alignment algorithm proceeds as follows. Let (p_1, p_2) denote a pair of points having the greatest distance in P , that is, a *diametrical pair*. For each distinct pair (q_1, q_2) in Q , the algorithm computes a rigid transformation matching (p_1, p_2) with (q_1, q_2) as follows. First, it applies to P a translation that maps p_1 to q_1 . Then it performs a rotation about p_1 (after translation) that aligns the directed line segment $\overrightarrow{p_1 p_2}$ with $\overrightarrow{q_1 q_2}$. The rigid transformation E resulting from the composition of this translation and rotation is then applied to the entire pattern set P and the Hausdorff distance $h(E(P), Q)$ is computed. After repeating this for all pairs (q_1, q_2) , the transformation with the smallest Hausdorff distance is returned. The running time of this algorithm is $O(n^2 m \log n)$ because, for each of the $n(n - 1)$ distinct pairs of Q , we compute the aligning transformation E in $O(1)$ time, and then, for each point $p \in P$, we compute the distance from $E(p)$ to its nearest neighbor in Q . (As in [17], if some a priori bound is provided for the optimum Hausdorff distance, it is possible to reduce the number of pairs (q_1, q_2) that need to be considered.) Nearest neighbors queries in a planar set Q can be answered in time $O(\log n)$ after $O(n \log n)$ preprocessing [7]. Because the analysis of our improved algorithm is closely related, let us review the proof of the approximation ratio given by Goodrich et al. [13]. Throughout, we let h_{opt} denote the optimum Hausdorff distance from P to Q .

Lemma 2.1 (Goodrich, Mitchell, and Orletsky) *The approximation ratio for serial alignment algorithm satisfies $A_{\text{ser}} \leq 4$.*

Proof It is easy to see that the serial alignment algorithm is invariant to the initial placement of P in the sense that if P and P' are equal up to a rigid transformation, then (assuming general position) the algorithm will map both these point sets to the same final positions with respect to Q and with the same Hausdorff distance. Thus to simplify matters, we may assume that P has been presented to the algorithm in its optimal position with respect to Q . Thus, for each $p_i \in P$, if we let q_i denote its closest point of Q then $\|p_i q_i\| \leq h_{\text{opt}}$. Now, run the serial alignment algorithm. We will bound the maximum distance by which any point of P has been displaced relative to its original optimal position, and this will provide the approximation ratio.

First, consider the effect of translation. Since p_1 is within distance h_{opt} of q_1 , each point of P is translated by a distance of at most h_{opt} , and so is now within distance $h_{\text{opt}} + h_{\text{opt}} = 2h_{\text{opt}}$ of its closest point of Q . Next, consider the effect of rotation. After translation we have $\|p_2 q_2\| \leq 2h_{\text{opt}}$, from which it follows that the rotation displaces p_2 by an additional distance of at most $2h_{\text{opt}}$. Since p_2 is the farthest point of P from p_1 , every other point of P is displaced through rotation by at most this distance. We see that after translation and rotation each point of P is at distance at most $2h_{\text{opt}} + 2h_{\text{opt}} = 4h_{\text{opt}}$ from its closest point of Q . Therefore the approximation bound is at most $4h_{\text{opt}}/h_{\text{opt}} = 4$.

(Observe further that after running the algorithm p_1 coincides with q_1 . Also, since rotation can only decrease the distance between p_2 and q_2 , after serial alignment has completed the distance from p_2 to q_2 is at most $2h_{\text{opt}}$. These observations will be used below in Lemma 2.3.) \square

There are two obvious shortcomings with this algorithm and its analysis. The first is that the algorithm does not optimally align the pair (p_1, p_2) with the pair (q_1, q_2)

with respect to Hausdorff distance. One would expect such an optimal alignment to provide a better approximation ratio. This observation is the basis of our algorithm. The second shortcoming is that the use of the triangle inequality in summing the two displacements neglects the presence of any geometrical dependence between these two vector quantities.

We present a new approach called *symmetric alignment*. The algorithm differs only in how the aligning transformation is computed. Let the pairs (p_1, p_2) in P and (q_1, q_2) in Q be chosen in the same way as they are in the serial alignment algorithm. Let m_p and m_q denote the respective midpoints of line segments $\overline{p_1 p_2}$ and $\overline{q_1 q_2}$. First, translate P to map m_p to m_q and then rotate P about m_p to align the directed segments $\overrightarrow{p_1 p_2}$ with $\overrightarrow{q_1 q_2}$. Thus, the only difference is that we align and rotate around the midpoints. It is easy to see that this alignment transformation minimizes the Hausdorff distance between the pairs (p_1, p_2) and (q_1, q_2) .

There is a pathological setting in which symmetric alignment can perform arbitrarily poorly compared to the optimum. This happens when, in the optimal Hausdorff placement of P , both p_1 and p_2 share the same closest point of Q . The problem is that symmetric alignment assumes that q_1 and q_2 are distinct (for otherwise the rotation angle is undefined). For this to occur the distance between p_1 and p_2 (that is, the diameter of P) must be less than the distance between the closest pair of points of Q . In such a case, the optimal placement results by simply aligning the center of the smallest enclosing disk for P with any point of Q . By standard methods in computational geometry (see [7]) this situation can be detected and handled in $O(m + n \log n)$ time, and so henceforth we will assume that it is not an issue.

Before giving our detailed analysis of the approximation ratio we establish a crude bound on the approximation ratio for symmetric alignment, which justifies the benefit of this approach over serial alignment.

Lemma 2.2 $A_{\text{sym}} \leq 2 + \sqrt{3} \approx 3.732$.

Proof As in the proof of Lemma 2.1, let us assume that P has been presented to the algorithm in its optimal position with respect to Q . Thus, each point of P is within distance h_{opt} of its closest point of Q . Recall that q_1 and q_2 denote the closest points of Q to p_1 and p_2 , respectively. For $i \in \{1, 2\}$ let \vec{v}_i denote the vector $\overrightarrow{p_i q_i}$.

First, let us consider translation. To match the midpoints m_p and m_q , we translate the pattern set P by the vector $\vec{t} = (\vec{v}_1 + \vec{v}_2)/2$. Since both of these points are within distance h_{opt} of their closest point of Q , we have $\|\vec{t}\| \leq h_{\text{opt}}$. Next we consider rotation. The distance between the translated point $p_1 + \vec{t}$ and q_1 is

$$\left\| \vec{v}_1 - \frac{\vec{v}_1 + \vec{v}_2}{2} \right\| = \left\| \frac{\vec{v}_1 - \vec{v}_2}{2} \right\| \leq h_{\text{opt}}.$$

Because (p_1, p_2) is a diametrical pair, it follows that all the points of P lie in a lune defined by the intersection of two discs both of radius $\|p_1 p_2\|$ centered at each of these points. Thus, no point of P is farther from the center of rotation (m_p) than the apex of this lune, which is at distance $(\sqrt{3}/2)\|p_1 p_2\|$. Since this rotation moves p_1 by a distance of at most h_{opt} , and p_1 is within distance $(1/2)\|p_1 p_2\|$ of m_p , it follows

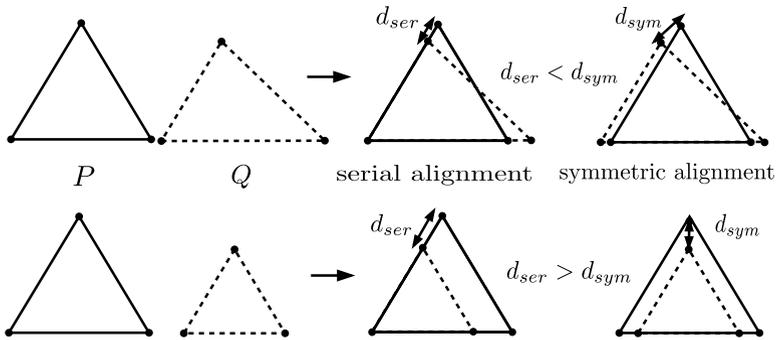


Fig. 1 Comparison of the two algorithms

that every point $p \in P$ is moved through rotation by an additional distance of at most $\sqrt{3}h_{\text{opt}}$.

Thus, we see that each point of P started within distance h_{opt} of its closest point of Q , and it has been displaced through translation and rotation by distances of at most h_{opt} and $\sqrt{3}h_{\text{opt}}$, respectively. Therefore, after symmetric alignment each point of P is within distance

$$h_{\text{opt}} + h_{\text{opt}} + \sqrt{3}h_{\text{opt}} = (2 + \sqrt{3})h_{\text{opt}} \approx 3.732 \cdot h_{\text{opt}},$$

of its closest point of Q . The approximation ratio follows immediately.

(Further observe that the points p_1 and p_2 started out within distance h_{opt} from their closest points of Q , and after alignment this is still true. This fact will be used below in Lemma 2.3.) □

Even though this crude approximation ratio is already an improvement, it suffers from the same problem as the earlier analysis in that it does not consider the geometric relationship between the translation and rotation vectors. Note that this does not imply that symmetric alignment is better than serial alignment for all input instances. In Fig. 1 we show two informal examples. In the first case serial alignment is better and in the second symmetric alignment is better. The following lemma shows, nonetheless, that the two methods achieve comparable approximation ratios on all input instances.

Lemma 2.3 *Given two planar point sets P and Q , let d_{sym} and d_{ser} denote the Hausdorff distances between P and Q after running symmetric alignment and serial alignment, respectively. Then*

$$d_{\text{sym}} \leq d_{\text{ser}} + \min\left(\frac{d_{\text{ser}}}{2}, h_{\text{opt}}\right) \leq \frac{3}{2}d_{\text{ser}} \quad \text{and}$$

$$d_{\text{ser}} \leq d_{\text{sym}} + h_{\text{opt}} \leq 2d_{\text{sym}}.$$

Proof Let us assume that P has been presented to the symmetric alignment algorithm in the position output by the serial alignment algorithm. Recall that (p_1, p_2)

is a diametrical pair from P , and (q_1, q_2) is the corresponding pair from Q . From the comments made at the end of the proof of Lemma 2.1 we know that p_1 and q_1 coincide, and the distance between p_2 and q_2 is at most $2h_{\text{opt}}$. Clearly, the distance between p_2 and q_2 is also at most d_{ser} . Now, we apply symmetric alignment. In order to align the midpoints of $\overline{p_1 p_2}$ with $\overline{q_1 q_2}$, each point of P is translated by the distance $\|p_2 q_2\|/2$, which is at most $\min(d_{\text{ser}}, 2h_{\text{opt}})/2$. No rotation is needed. Therefore, we have

$$d_{\text{sym}} \leq d_{\text{ser}} + \frac{\|p_2 q_2\|}{2} \leq d_{\text{ser}} + \min\left(\frac{d_{\text{ser}}}{2}, h_{\text{opt}}\right) \leq \frac{3}{2}d_{\text{ser}}.$$

To prove the other bound, let us assume that the point set P has been presented to the serial alignment algorithm in the positions output by the symmetric alignment algorithm. As observed at the end of the proof of Lemma 2.2, $\|p_1 q_1\| \leq h_{\text{opt}}$. If we apply serial alignment to these point sets, each point of P will be translated by at most $\|p_1 q_1\|$. No rotation is needed. Therefore,

$$d_{\text{ser}} \leq d_{\text{sym}} + \|p_1 q_1\| \leq d_{\text{sym}} + h_{\text{opt}} \leq 2d_{\text{sym}}. \quad \square$$

3 Symmetric Alignment: Upper Bound

In this section we derive an upper bound on the approximation ratio for symmetric alignment. As before let h_{opt} denote the optimum Hausdorff distance between P and Q achievable under any rigid transformation of P . As mentioned above, our analysis is sensitive to a geometric parameter that is proportional to the ratio of P 's diameter to the optimum Hausdorff distance. Define the *distance ratio* to be

$$\rho = \frac{\text{diam}(P)}{2h_{\text{opt}}},$$

where $\text{diam}(P)$ denotes diameter of P . Clearly $h_{\text{opt}} \leq \text{diam}(P)$ and therefore $\rho \geq \frac{1}{2}$. Recall that $A_{\text{sym}}(\rho)$ denotes the approximation ratio for symmetric alignment as a function of the distance ratio. The main result of this section is presented next.

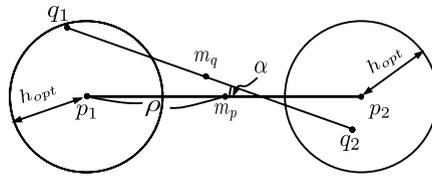
Theorem 1 Consider two planar point sets P and Q , and let ρ be the distance ratio for P and Q . Then

$$A_{\text{sym}}(\rho) \leq \min\left(3 + \frac{1}{\sqrt{3}\rho}, \sqrt{4\rho^2 + 2\rho + 1}\right),$$

and further $A_{\text{sym}}(\rho) \leq 3.14$ for all ρ .

Recall that the distance ratio is large in many applications. This theorem shows that as ρ tends to infinity, the approximation ratio is at most 3. The remainder of this section is devoted to proving this theorem. As usual, it will simplify the presentation to assume that P has been transformed to its optimal placement with respect to Q , and we will bound the amount of additional displacement of each point of P relative

Fig. 2 The positions of the point sets prior to running the algorithm



to this placement. Without loss of generality we may scale space uniformly so that the optimum Hausdorff distance h_{opt} is 1. Recall that (p_1, p_2) is a diametrical pair from P , and (q_1, q_2) is the corresponding pair from Q . From our scaling it follows that for $i \in \{1, 2\}$, q_i lies within a disc of radius 1 centered at p_i . (See Fig. 2.) Let m_p and m_q denote the respective midpoints of the segments $\overline{p_1p_2}$ and $\overline{q_1q_2}$. Note that ρ is just the distance from m_p to either p_1 or p_2 . Let $\alpha \geq 0$ denote the absolute acute angle between the lines supporting the segments $\overline{p_1q_1}$ and $\overline{p_2q_2}$. If $\rho > 1$, then the two unit discs do not intersect. Hence, after aligning the midpoints by translation, the distance from p_1 to q_1 is still at most $h_{\text{opt}} (= 1)$ (see Lemma 2.2), and the orthogonal distance from p_1 to the line $\overline{q_1q_2}$, which is $\rho \sin \alpha$, is less than or equal to the distance from p_1 to any point on the line. It follows therefore that $0 \leq \rho \sin \alpha \leq 1$.

Before presenting the details, we give a brief overview of our proof structure. We establish two upper bounds on the approximation ratio, each a function of ρ . One bound is better for small ρ and the other better for large ρ . We shall show by numerical computations that the two functions cross over one another at a value $\rho_{\text{sym}}^* \approx 1.26$ and that $A_{\text{sym}}(\rho_{\text{sym}}^*) \approx 3.14$. Furthermore, both these functions decrease monotonically as ρ moves away from ρ_{sym}^* . We begin by establishing the approximation ratio for low distance ratios.

Lemma 3.1 For all ρ , $A_{\text{sym}}(\rho) \leq \sqrt{4\rho^2 + 2\rho + 1}$.

Proof Let (p_1, p_2) and (q_1, q_2) be as defined in the algorithm. As mentioned in the proof of Lemma 2.2, every point of P lies within a lune formed by the intersection of two discs of radius 2ρ centered at p_1 and p_2 . Hence, every point of P is within distance $\sqrt{3}\rho$ of the midpoint m_p . After applying symmetric alignment, $\overline{p_1p_2}$ and $\overline{q_1q_2}$ are collinear and, for $i \in \{1, 2\}$, the distance $\|p_iq_i\|$ is at most $h_{\text{opt}} (= 1)$. Since, $\|p_im_p\| = \rho$, it follows directly that every point of P is within distance $\sqrt{(\rho + 1)^2 + (\sqrt{3}\rho)^2}$ of either q_1 or q_2 . Simplifying yields the desired bound. \square

The above approximation ratio is very crude as it considers no points of Q other than q_1 and q_2 . Hence, it is only useful for very small values of ρ . To deal with the case of high ρ values we consider the effects of translation and rotation separately. We will characterize the set of possible translations that align m_p with m_q as a function of ρ and the rotation angle α . The distance by which an arbitrary point of $p \in P \setminus \{p_1, p_2\}$ can be displaced by the rotation is not only a function of these two parameters, but is also a function of the distance of this point from the center of rotation m_p . We will then bound the sum of these two displacement distances. These translation and rotation analyses are presented in Sects. 3.1 and 3.2, respectively, and these results are combined in Sect. 3.3.

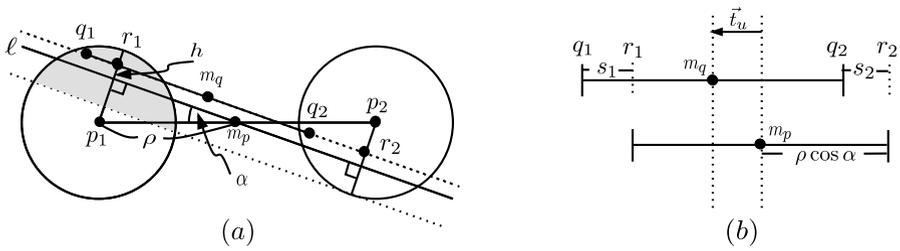


Fig. 3 Analysis of the midpoint translation

3.1 Translational Displacement

We begin by considering the space of possible translations that align m_p with m_q . It will make matters a bit simpler to think of translating Q to align m_q with m_p , but of course any bounds on the distance of translation will apply to case of translating P . The translation is given by a vector which we denote by \vec{t} . The following lemma bounds the length of this translation vector as a function of ρ and α .

Lemma 3.2 *The symmetric alignment’s translation transformation displaces any point of P by a vector \vec{t} of length*

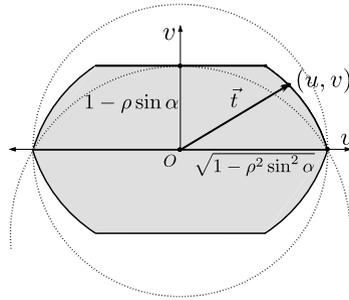
$$\|\vec{t}\| \leq \sqrt{1 - \rho^2 \sin^2 \alpha}.$$

Proof Recall that (p_1, p_2) and (q_1, q_2) denote the respective point pairs from the sets P and Q and that α is the acute angle between the lines $\overline{p_1p_2}$ and $\overline{q_1q_2}$. Since we have assumed that P is placed in the optimal position, q_1 and q_2 must lie inside the circles of radius $h_{\text{opt}} (= 1)$ centered at p_1 and p_2 , respectively. For the sake of illustration let us assume that $\overline{p_1p_2}$ is directed horizontally from left to right and that $\overline{q_1q_2}$ has a nonpositive slope. (See Fig. 3(a).)

Consider a line passing through m_p that is parallel to $\overline{q_1q_2}$. Let r_1 and r_2 be the respective orthogonal projections of p_1 and p_2 onto the line $\overline{q_1q_2}$, and let s_1 and s_2 denote the respective signed displacements $\|\overline{r_1q_1}\|$ and $\|\overline{r_2q_2}\|$ along this line (See Fig. 3(b).) Consider a coordinate system centered at m_p whose positive u -axis is located along a line ℓ that is directed from left to right, and whose positive v -axis is perpendicular to this and directed upwards. In this coordinate system we have $m_p = (0, 0)$, $q_1 = (s_1 - \rho \cos \alpha, h)$, and $q_2 = (s_2 + \rho \cos \alpha, h)$. Thus, $m_q = ((s_1 + s_2)/2, h)$, and the translation vector \vec{t} is $m_q - m_p = m_q$. By straightforward calculations we have $|s_1| \leq \sqrt{1 - (\rho \sin \alpha + h)^2}$ and $|s_2| \leq \sqrt{1 - (\rho \sin \alpha - h)^2}$. Therefore,

$$\begin{aligned} \|\vec{t}\|^2 &= u^2 + v^2 = \left(\frac{s_1 + s_2}{2}\right)^2 + h^2 = \frac{1}{2}(s_1^2 + s_2^2) - \frac{1}{4}(s_1 - s_2)^2 + h^2 \\ &\leq \frac{1}{2}(s_1^2 + s_2^2) + h^2 \leq \frac{1}{2}[(1 - (\rho \sin \alpha + h)^2) + (1 - (\rho \sin \alpha - h)^2)] + h^2 \\ &= 1 - \rho^2 \sin^2 \alpha. \end{aligned}$$

Fig. 4 Translation space, $\mathcal{T}_\rho(\alpha)$, for symmetric alignment



Observe that the translation is maximized when $s_1 = s_2$, and this implies that the line $\overline{q_1q_2}$ passes through m_p . (This will be used in our lower bound construction in Sect.4.) □

Given α and ρ , let us refer to the set of all valid (u, v) translation vectors as the *translation space*, denoted $\mathcal{T}_\rho(\alpha)$. A more detailed analysis of the set of valid (u, v) pairs shows that the set of possible translational displacements lies within a region of the (u, v) plane as illustrated in the shaded region of Fig. 4, but we will not need this for our subsequent analysis.

3.2 Rotational Displacement

Next we consider the effect of rotation on the approximation error. Unlike translation we consider the placement of points in P because the distance by which a point is displaced by rotation is determined by the distance to the center of rotation. For each point $p \in P$, let \vec{r}_p denote its displacement due to rotation. As mentioned in the proof of Lemma 2.2 every point of P lies within a lune formed by the intersection of two discs of radius 2ρ centered at p_1 and p_2 . (See Fig. 5(a).) The following lemma describes the possible displacements of a point of P under the rotational part of the aligning transformation. This is done relative to a coordinate system centered at m_p , whose x -axis is directed towards p_2 .

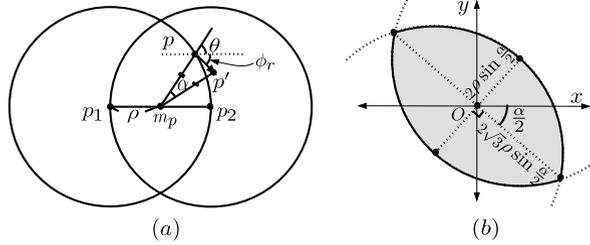
Lemma 3.3 *The symmetric alignment’s rotation transformation displaces any point $p \in P$ by a vector \vec{r}_p of length*

$$\|\vec{r}_p\| \leq 2\sqrt{3}\rho \sin \frac{\alpha}{2}.$$

Proof Let θ be the signed angle from $\overline{m_p p_2}$ to $\overline{m_p p}$. We may assume $-\pi \leq \theta \leq \pi$. Let p' denote the point after rotating p clockwise about m_p by angle α . (See Fig. 5(a).)

By simple trigonometry and the observation that $\Delta m_p p p'$ is isosceles, it follows that the displacement vector $\overline{p p'}$, length $\|\vec{r}_p\| = \|m_p p\| \cdot 2 \sin \frac{\alpha}{2}$. The farthest point of the lune from m_p is easily seen to be the apex, which is at distance $\sqrt{3}\rho$. Thus, $\|\vec{r}_p\|$ is at most $2\sqrt{3}\rho \sin \frac{\alpha}{2}$. □

Fig. 5 Analysis of the rotational displacements for symmetric alignment



As we did for translation, for a given pair of parameter values α and θ we can define the set of possible vector displacements for all points $p \in P$, which we call the *rotation space*, denoted $\mathcal{R}_\rho(\alpha)$. This is illustrated in Fig. 5(b), but is not needed for the rest of our analysis. Henceforth, when p is clear for context, we will refer to \vec{r}_p as \vec{r} .

3.3 Combining Translation and Rotation

We are now ready to derive the approximation ratio for the symmetric alignment algorithm by combining the bounds on the translational and rotational displacements. Recall that we apply Lemma 3.1 to bound $A_{\text{sym}}(\rho)$ for small values of ρ , and so here we will assume that $\rho \geq \rho_{\text{sym}}^*$.

Recall that at the start of the algorithm, the points are assumed to be placed in the optimal positions and that space has been scaled so that $h_{\text{opt}} = 1$. Let us fix any point $p \in P$, let \vec{t} denote its displacement due to translation, and let $\vec{r} = \vec{r}_p$ denote its displacement due to rotation. It follows that p has been displaced from its initial position by a distance of $\|\vec{t} + \vec{r}\|$. Since its initial position was within distance $h_{\text{opt}} (= 1)$ of some point of Q , it is now within distance $\|\vec{t} + \vec{r}\| + 1$ of some point of Q . Recalling that $\mathcal{T}_\rho(\alpha)$ and $\mathcal{R}_\rho(\alpha)$ denote the space of possible translational and rotational displacement vectors and that $0 \leq \sin \alpha \leq 1/\rho$, we have

$$A_{\text{sym}}(\rho) \leq \max_{\alpha} \left(\max_{\substack{\vec{t} \in \mathcal{T}_\rho(\alpha) \\ \vec{r} \in \mathcal{R}_\rho(\alpha)}} \|\vec{t} + \vec{r}\| \right) + 1.$$

Unfortunately, determining this length bound exactly would involve solving a relatively high order equation involving trigonometric functions. Instead, we will simplify matters by applying the triangle inequality to separate the translational and rotational components, which are in turn bounded in Lemmas 3.2 and 3.3.

$$\begin{aligned} A_{\text{sym}}(\rho) &\leq \max_{\alpha} \left(\max_{\vec{t} \in \mathcal{T}_\rho(\alpha)} \|\vec{t}\| + \max_{\vec{r} \in \mathcal{R}_\rho(\alpha)} \|\vec{r}\| \right) + 1 \\ &\leq \max_{\alpha} \left(\sqrt{1 - \rho^2 \sin^2 \alpha} + 2\sqrt{3}\rho \sin \frac{\alpha}{2} \right) + 1. \end{aligned}$$

Substituting $x = \cos \alpha$, it follows that the quantity to be maximized is

$$f_\rho(x) = \sqrt{1 - \rho^2(1 - x^2)} + \sqrt{6}\rho\sqrt{1 - x} + 1, \quad \text{where } \sqrt{1 - \frac{1}{\rho^2}} \leq x \leq 1.$$

To determine the maximum value of f_ρ we take the partial derivative with respect to x , yielding

$$\frac{\partial f_\rho}{\partial x} = -\frac{\sqrt{3}\rho}{\sqrt{2(1-x)}} + \frac{\rho^2 x}{\sqrt{1 - \rho^2(1 - x^2)}}.$$

By our assumption that $\rho \geq \rho_{\text{sym}}^* \approx 1.26$, it follows by symbolic manipulations that $\partial f_\rho / \partial x = 0$ has a single real root for $\rho \geq \rho_{\text{sym}}^*$, which is

$$x_0(\rho) = \frac{1}{6} \left(-1 + c_1(\rho) + \frac{1}{c_1(\rho)} \right),$$

where

$$c_1(\rho) = \frac{\rho^2}{(161\rho^6 + 18\rho^4\sqrt{c_2(\rho)} - 162\rho^4)^{1/3}} \quad \text{and}$$

$$c_2(\rho) = 80\rho^4 - 161\rho^2 + 81.$$

By an analysis of this derivative it follows that the function f_ρ achieves its maximum value when $x = x_0(\rho)$. Thus, we have the following.

Lemma 3.4 *For $\rho \geq \rho_{\text{sym}}^* \approx 1.26$ we have*

$$A_{\text{sym}}(\rho) \leq f_\rho(x_0(\rho)) = \sqrt{1 - \rho^2(1 - x_0(\rho)^2)} + \sqrt{6}\rho\sqrt{1 - x_0(\rho)} + 1,$$

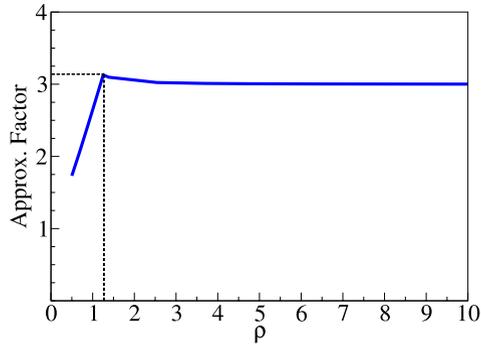
where $x_0(\rho)$ is defined above.

Unfortunately, this function is too complex to reason about easily. Nonetheless, we can evaluate it numerically for any fixed value of ρ . The resulting approximation ratio (together with the alternate approximation ratio of Lemma 3.1) is plotted as a function of ρ in Fig. 6. A numerical analysis shows that the two functions cross over at $\rho_{\text{sym}}^* \approx 1.26$, and at this point the function achieves its maximum value of roughly 3.14. The figure also indicates that the approximation ratio approaches 3 as ρ tends to ∞ . The following result establishes this asymptotic convergence by proving a somewhat weaker approximation ratio.

Lemma 3.5 *If $\rho > \rho_{\text{sym}}^*$ then $A_{\text{sym}}(\rho) \leq 3 + \frac{1}{\sqrt{3}\rho}$.*

Proof Before presenting the general case, we consider the simpler limiting case when ρ tends to ∞ . Since $0 \leq \sin \alpha \leq 1/\rho$, in the limit α approaches 0. Using the fact of

Fig. 6 The approximation ratio for the symmetric alignment algorithm as a function of ρ



$\lim_{\alpha \rightarrow 0} \frac{\sin \alpha}{\alpha} = 1$ we have

$$\begin{aligned} A_{\text{sym}}(\infty) &= \lim_{\rho \rightarrow \infty} A_{\text{sym}}(\rho) \leq \lim_{\rho \rightarrow \infty} \sqrt{1 - \rho^2 \sin^2 \alpha} + 2\sqrt{3}\rho \sin \frac{\alpha}{2} + 1 \\ &= \lim_{\rho \rightarrow \infty} \sqrt{1 - \rho^2 \alpha^2} + \sqrt{3}\rho \alpha + 1. \end{aligned}$$

Let $x = \rho\alpha$. In the limit we have $0 \leq x \leq 1$ and so

$$A_{\text{sym}}(\infty) \leq \max_{0 \leq x \leq 1} \left(\sqrt{1 - x^2} + \sqrt{3}x + 1 \right).$$

It is easy to verify that $A_{\text{sym}}(\infty)$ achieves a maximum value of 3 when $x = \sqrt{3}/2$.

Next, we consider the general case. To simplify $A_{\text{sym}}(\rho)$, we apply a Taylor’s series expansion.

$$\begin{aligned} A_{\text{sym}}(\rho) &\leq \max_{\alpha} \left(\sqrt{1 - \rho^2 \sin^2 \alpha} + 2\sqrt{3}\rho \sin \frac{\alpha}{2} + 1 \right) \\ &\leq \max_{\alpha} \left(\sqrt{1 - \rho^2 \left(\alpha - \frac{\alpha^3}{6} \right)^2} + \sqrt{3}\rho \alpha + 1 \right). \end{aligned} \tag{1}$$

Since $0 \leq \alpha \leq \arcsin(1/\rho)$, we have $0 \leq \rho\alpha \leq \rho \arcsin(1/\rho)$. For all $\rho > \rho_{\text{sym}}^*$, observe that $\rho \arcsin(1/\rho)$ is at most $\rho_{\text{sym}}^* \arcsin(1/\rho_{\text{sym}}^*)$. Let us denote the value by c_0 , which by numerical evaluation is approximately 1.16. Thus, we have $0 \leq \rho\alpha < c_0$. The rest of the analysis is broken into two cases: $0 \leq \rho\alpha \leq 1$ and $1 < \rho\alpha < c_0$.

In the first case ($0 \leq \rho\alpha \leq 1$) observe that $1 - \rho^2 \alpha^2 \geq 0$. We argued above that $(\sqrt{1 - \rho^2 \alpha^2} + \sqrt{3}\rho\alpha + 1) \leq 3$ and so

$$\begin{aligned} A_{\text{sym}}(\rho) &\leq \max_{\alpha} \left(\sqrt{1 - \rho^2(\alpha - \alpha^3/6)^2} + \sqrt{3}\rho\alpha + 1 \right. \\ &\quad \left. + \left[3 - (\sqrt{1 - \rho^2 \alpha^2} + \sqrt{3}\rho\alpha + 1) \right] \right) \\ &= \max_{\alpha} \left(\sqrt{1 - \rho^2(\alpha - \alpha^3/6)^2} - \sqrt{1 - \rho^2 \alpha^2} + 3 \right). \end{aligned}$$

To simplify this let $x = \rho\alpha$. We see that $A_{\text{sym}}(\rho) \leq \max_{0 \leq x \leq 1} g(x)$ where,

$$g(x) = \sqrt{1 - x^2 \left(1 - \frac{x^2}{6\rho^2}\right)^2} - \sqrt{1 - x^2} + 3.$$

For all fixed ρ this is a monotonically increasing function in x . Since $x \leq 1$, it is easy to verify that this function achieves its maximum value at $x = 1$. Therefore,

$$A_{\text{sym}}(\rho) \leq g(1) = 3 + \sqrt{1 - \left(1 - \frac{1}{6\rho^2}\right)^2} = 3 + \sqrt{\frac{1}{3\rho^2} - \frac{1}{36\rho^4}} \leq 3 + \frac{1}{\sqrt{3}\rho},$$

as desired.

For the second case ($1 < \rho\alpha < c_0$) we cannot use the above approach because $1 - \rho^2\alpha^2 < 0$. Nonetheless, by expanding the first term of (1) we have

$$\begin{aligned} \sqrt{1 - \rho^2 \sin^2 \alpha} &\leq \sqrt{1 - \rho^2(\alpha - \alpha^3/6)^2} = \sqrt{(1 - \rho^2\alpha^2) + \frac{1}{3}\rho^2\alpha^4 - \frac{1}{36}\rho^2\alpha^6} \\ &< \sqrt{\frac{1}{3}\rho^2\alpha^4} = \frac{1}{\sqrt{3}}\rho\alpha^2. \end{aligned}$$

Therefore,

$$A_{\text{sym}}(\rho) \leq \max_{\alpha} \left(\frac{1}{\sqrt{3}}\rho\alpha^2 + \sqrt{3}\rho\alpha + 1 \right).$$

It is straightforward to verify that the above function achieves its maximum value when $\rho\alpha = c_0$, and this value is at most $3 + 1/(\sqrt{3}\rho)$. □

Combining this with Lemma 3.1 and the previous comments about the cross-over point completes the proof of Theorem 1.

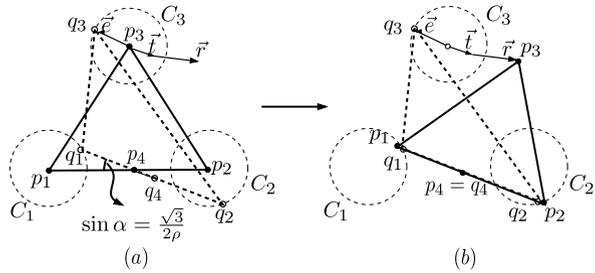
4 Symmetric Alignment: Lower Bound

It is natural to wonder whether the upper bound on the approximation ratio A_{sym} proved in Theorem 1 is tight. We show that this is nearly the case. In particular, we show that for all sufficiently large ρ the approximation factor is strictly greater than 3, and it approaches 3 in the limit.

Theorem 2 *For all sufficiently large ρ , there exists an input on which symmetric alignment achieves an approximation ratio of at least $3 + \frac{1}{10\rho^2}$.*

The remainder of this section is devoted to proving this. Our approach is to consider a configuration of points that generates the worst case (that is, the largest translation and rotational displacements) for Lemmas 3.2 and 3.3. Consider a fixed value of ρ . We define the pattern point set $P = \{p_1, p_2, p_3, p_4\}$ as follows. The first three points form an equilateral triangle of side length 2ρ oriented in counterclockwise

Fig. 7 The lower bound on A_{sym}



order. We place p_4 at the midpoint of $\overline{p_1 p_2}$. (See Fig. 7(a).) By an infinitesimal perturbation of the points, we may assume that the pair (p_1, p_2) is the unique diametrical pair for P .

We define the background set $Q = \{q_1, q_2, q_3, q_4\}$ so that the optimum Hausdorff distance will be at most 1. For $1 \leq i \leq 4$, let C_i be a circle of unit radius centered p_i . Consider an angle $\alpha > 0$ such that $\sin \alpha = \frac{\sqrt{3}}{2\rho}$ and construct a line passing through p_4 (the midpoint of $\overline{p_1 p_2}$) forming an angle α with line $\overline{p_1 p_2}$. Place q_1 and q_2 at the rightmost intersection points of this line and C_1 and C_2 , respectively. (See Fig. 7(a).) Running symmetric alignment so that (p_1, p_2) is aligned with (q_1, q_2) results in a translation and rotation. Let \vec{t} and \vec{r} denote the translation and rotational displacement vectors for p_3 , respectively. Let $q_3 = p_3 + \vec{e}$, where \vec{e} is a vector of unit length whose direction is chosen to be directly opposite that of $\vec{t} + \vec{r}$. Intuitively, q_3 has been chosen to be as far away as possible from p_3 after alignment. Finally, place q_4 at the midpoint of $\overline{q_1 q_2}$. Observe that prior to alignment, each p_i is within distance 1 from q_i , and therefore $h_{\text{opt}} \leq 1$.

Consider the placement of P and Q after symmetric alignment. (See Fig. 7(b).) We will analyze the displacement distance of p_3 relative to q_3 . Because of the asymmetry introduced by p_4 and q_4 , it is intuitively clear that if the distance ratio is high enough, symmetric alignment will achieve the best match by aligning the pair (q_1, q_2) with (p_1, p_2) . We establish this formally in the next lemma.

Lemma 4.1 *For all sufficiently large ρ , symmetric alignment aligns (p_1, p_2) with (q_1, q_2) .*

Proof Assume that $\rho > 2A_{\text{sym}}$. We will show that the choice of any other alignment pair will result in a Hausdorff distance greater than $A_{\text{sym}}h_{\text{opt}}$, a contradiction. Suppose that the algorithm aligns (p_1, p_2) with some pair (q_i, q_j) , where $(q_i, q_j) \neq (q_1, q_2)$. For $1 \leq i \leq 4$, let C_i denote the circle of radius A_{sym} centered at point p_i , after alignment. (These are different from the circles C_i used in the construction.) Since $\rho > 2A_{\text{sym}}$, all these circles are disjoint. In order for the Hausdorff distance to be less than A_{sym} , after alignment each of these circles must contain exactly one point of Q .

First, we will show that q_4 should be in C_4 . Suppose that to the contrary that q_4 is in circle C_k , $k \neq 4$. Then q_1 must lie in some other circle. Since q_1 and q_2 are symmetric with respect to the point q_4 , the point q_2 cannot be in any circle, a contradiction.

From the hypothesis that $(p_1, p_2) \neq (q_1, q_2)$, either q_1 or q_2 should be in C_3 . Let us assume that q_1 is in C_3 . (The other case is similar.) Then q_2 may be in either C_1 or C_2 . However, if q_2 lies in C_2 , then clearly q_3 lies in none of the circles. (Note that we do not allow for reflection.) Thus, q_2 must lie within C_1 and q_3 lies within C_2 , implying that (q_2, q_3) is the chosen aligning pair for (p_1, p_2) . Thus, (after alignment) the midpoint of $\overline{q_2q_3}$, called it m_{23} , coincides with p_4 . Since $\triangle q_4q_2m_{23}$ and $\triangle q_1q_2q_3$ are similar up to a scale factor of 2, we have

$$\|p_4q_4\| = \frac{\|q_1q_3\|}{2}.$$

Also, observe that prior to alignment, each point of Q is within distance h_{opt} of its corresponding point of P , and so by the triangle inequality the distance between any two points $\|q_iq_j\|$ of Q is within $2h_{\text{opt}}$ of the distance $\|p_ip_j\|$. So $\|q_1q_3\| \geq \|p_1p_3\| - 2h_{\text{opt}}$. Since the transformation is rigid, this is true after alignment. Since $\|p_1p_3\| = 2\rho$, we have

$$\|p_4q_4\| \geq \frac{\|p_1p_3\| - 2h_{\text{opt}}}{2} = \frac{2\rho - 2h_{\text{opt}}}{2} = \rho - h_{\text{opt}}.$$

This exceeds $A_{\text{sym}}h_{\text{opt}}$ for all sufficiently large ρ , yielding the desired contradiction. □

To complete the analysis of the lower bound, let us consider the translational and rotational displacement distances of p_3 that result from symmetric alignment. The line segment $\overline{q_1q_2}$ passes through p_4 (the midpoint of $\overline{p_1p_2}$). Let $\overline{q'_2q_2}$ denote the intersection of the line segment $\overline{q_1q_2}$ with the circle C_2 . By simple trigonometry $\|q'_2q_2\| = 2\sqrt{1 - \rho^2 \sin^2 \alpha}$, which equals 1 by our choice of α . It is easy to see that the translation distance is half the chord length $\|q'_2q_2\|$ along a direction at angle $-\alpha$. Thus the translational displacement vector is

$$\vec{t} = \frac{1}{2} \|q_2q'_2\| (\cos(-\alpha), \sin(-\alpha)) = \frac{1}{2} (\cos(-\alpha), \sin(-\alpha)).$$

Next, let us consider the displacement of p_3 due to rotation. The rotation is about p_4 , the midpoint of p_1 and p_2 , and the angle of rotation is α . Since $\triangle p_1p_2p_3$ is an equilateral triangle of side length 2ρ , the distance from the center of rotation to p_3 is $\sqrt{3}\rho$. From simple trigonometry, the displacement distance due to rotation is $\sqrt{3}\rho \cdot 2 \sin \frac{\alpha}{2}$ and the direction of the displacement is $-\frac{\alpha}{2}$. Thus the rotational displacement vector is

$$\vec{r} = 2\sqrt{3}\rho \sin \frac{\alpha}{2} \left(\cos \left(-\frac{\alpha}{2} \right), \sin \left(-\frac{\alpha}{2} \right) \right).$$

Observe that the angle between \vec{t} and \vec{r} is $\alpha/2$.

To complete the analysis, we decompose the rotational displacement vector \vec{r} into two components: \vec{r}_1 is parallel to \vec{t} and \vec{r}_2 is perpendicular to it. We have

$$\|\vec{r}_1\| = 2\sqrt{3}\rho \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = \sqrt{3}\rho \sin \alpha = \frac{3}{2},$$

$$\|\vec{r}_2\| = 2\sqrt{3}\rho \sin^2 \frac{\alpha}{2} = \sqrt{3}\rho(1 - \cos \alpha) = \sqrt{3}\rho \left(1 - \sqrt{1 - \frac{3}{4\rho^2}}\right).$$

The squared length of the displacement vector is $D = \|\vec{t} + \vec{r}_1 + \vec{r}_2\|^2 = (\|\vec{t}\| + \|\vec{r}_1\|)^2 + \|\vec{r}_2\|^2$. Thus, we have

$$D = \left(\frac{1}{2} + \frac{3}{2}\right)^2 + \|\vec{r}_2\|^2 = 4 + 3\rho^2 \left(1 - \sqrt{1 - \frac{3}{4\rho^2}}\right)^2.$$

Using the fact that $\sqrt{1-x} \leq 1 - \frac{x}{2}$ we have

$$D \geq 4 + 3\rho^2 \left(1 - \left(1 - \frac{3}{8\rho^2}\right)\right)^2 = 4 + \frac{27}{64\rho^2}.$$

We may assume that $\rho \geq 2$, from which it follows that

$$\frac{27}{64\rho^2} \geq \frac{4}{10\rho^2} + \frac{1}{400\rho^2} \geq \frac{4}{10\rho^2} + \frac{1}{100\rho^4}.$$

Thus, we obtain

$$D \geq 4 + \frac{4}{10\rho^2} + \frac{1}{100\rho^4} = \left(2 + \frac{1}{10\rho^2}\right)^2 \quad \text{and so} \quad \sqrt{D} \geq 2 + \frac{1}{10\rho^2}.$$

By adding the initial distance of 1 from q_3 to p_3 to the displacement, it follows that the final distance from p_3 to q_3 is $1 + \sqrt{D}$. Combining this with the fact that $h_{\text{opt}} \leq 1$, the approximation ratio satisfies

$$A_{\text{sym}}(\rho) \geq \frac{1 + \sqrt{D}}{h_{\text{opt}}} \geq 3 + \frac{1}{10\rho^2}.$$

This completes the proof of Theorem 2. Clearly this is greater than 3 for all positive ρ and converges to 3 in the limit as ρ tends to ∞ .

5 Serial Alignment: Upper Bound

In this section we derive an upper bound on the approximation ratio $A_{\text{ser}}(\rho)$ for serial alignment as a function of the distance ratio ρ . (Recall the definitions presented at the start of Sect. 3.) The main result of this section is presented below.

Theorem 3 Consider two planar point sets P and Q , and let ρ be the distance ratio for P and Q . Then

$$A_{\text{ser}}(\rho) \leq \min\left(c_\infty + \frac{9}{4\rho}, 2\rho\right),$$

where c_∞ is defined to be $\lim_{\rho \rightarrow \infty} A_{\text{ser}}(\rho) \approx 3.196$. Further $A_{\text{ser}}(\rho) \leq 3.44$ for all ρ .

As in the analysis of symmetric alignment we will establish two approximation ratios as functions of ρ , one is better for low ρ and the other for high ρ . We will show that the two functions cross over at a value $\rho_{\text{ser}}^* \approx 1.72$ and that $A_{\text{ser}}(\rho_{\text{ser}}^*) \approx 3.44$. Furthermore, both these functions decrease monotonically as ρ moves away from ρ_{ser}^* . The approximation ratio for low distance ratios is presented in Lemma 5.1 below. As usual, assume that P has been transformed to its optimal placement with respect to Q . Let (p_1, p_2) denote the diametrical pair in P and let (q_1, q_2) denote the corresponding pair Q . Let $\alpha \geq 0$ denote the absolute acute angle between the lines $\overline{p_1 p_2}$ and $\overline{q_1 q_2}$.

Lemma 5.1 For all ρ , $A_{\text{ser}}(\rho) \leq 2\rho$.

Proof After applying serial alignment p_1 and q_1 coincide. Thus, after alignment the distance between any point $p \in P$ and q_1 is at most 2ρ since the distance from p_1 to p is at most 2ρ . Therefore, the Hausdorff distance after alignment is at most 2ρ . \square

To deal with the more interesting case of high ρ we consider the effects of translation and rotation separately as functions of ρ and the rotation angle α . These results are presented Sects. 5.1 and 5.2, respectively. Unlike the corresponding analysis for symmetric alignment of Sect. 3, the analysis here will require a more detailed understanding of the geometric structure of the translation and rotation spaces.

5.1 Translational Displacement

Let us consider the space of possible translation vectors that align p_1 with q_1 . Since P is in its optimal placement, it follows that q_1 and q_2 lie within two circles C_1 and C_2 of radius $h_{\text{opt}} (= 1)$ centered at p_1 and p_2 , respectively. (See Fig. 8(a).) For the sake of illustration, let us assume that $\overline{p_1 p_2}$ is directed horizontally from left to right as shown in the figure and that the line $\overline{q_1 q_2}$ has a nonpositive slope (that is, α is a clockwise angle). Consider a line ℓ that is parallel to $\overline{q_1 q_2}$ and is tangent to C_2 from below. This line forms an angle of α with respect to $\overline{p_1 p_2}$ and clearly it intersects C_1 .

Let h_1 denote the orthogonal projection of p_1 onto the line ℓ , and let h_2 denote the orthogonal projection of p_2 onto the line passing through p_1 and parallel to ℓ . By simple trigonometry $\|p_2 h_2\| = \|p_1 p_2\| \sin \alpha$ and since C_2 's radius is 1, the signed

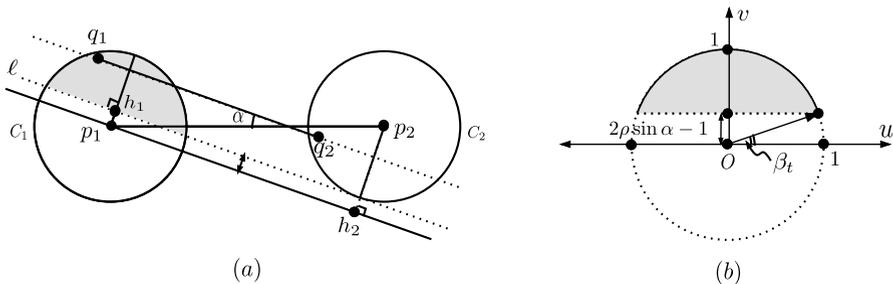


Fig. 8 The translation space for serial alignment

distance from p_1 to h_1 is $2\rho \sin \alpha - 1$. The interior of circle C_1 lying above ℓ defines a region called a *segment* of C_1 . (This is the shaded region of Fig. 8(a).) Clearly, q_1 lies within this segment. It follows that the set of possible translation vectors $\overrightarrow{p_1 q_1}$ is a copy of this region translated so that p_1 coincides with the origin.

As in Sect. 3.1, we introduce (u, v) coordinate system by rotating original coordinate system by α . In particular, the u -axis is parallel to $\overrightarrow{q_1 q_2}$, and the v -axis is orthogonal and directed upwards. This yields the following characterization of the translation space in terms of this coordinate system, which is illustrated in Fig. 8(b). In this section and the next it will simplify notation somewhat to use polar coordinates to represent vectors. The polar coordinates $\langle s, \phi \rangle$ indicate that s is the length of the vector, and ϕ is a counterclockwise angle with respect to the positive horizontal axis.

Lemma 5.2 *For fixed α and ρ , the translation space $\mathcal{T}_\rho(\alpha)$ in (u, v) coordinates is the segment of a circle of radius $h_{\text{opt}} (= 1)$ centered the origin that lies above the line $v = 2\rho \sin \alpha - 1$. The rightmost vertex of this segment has polar coordinates $\langle 1, \beta_t \rangle$, where $\sin \beta_t = 2\rho \sin \alpha - 1$.*

For now we assume that $\beta_t > 0$. Later in Lemma 5.6 we will show that this assumption is safe, since if $\beta_t \leq 0$ the resulting bound will be weaker than for positive β_t .

5.2 Rotational Displacement

Next we consider the effects of rotation on the approximation error. As in Sect. 3.2 we need to consider the placement of points in P because the distance by which a point is displaced by rotation depends on the distance from this point to the center of rotation. For given ρ and α values, let $\mathcal{R}_\rho(\alpha)$ denote the set of possible rotational displacement vectors.

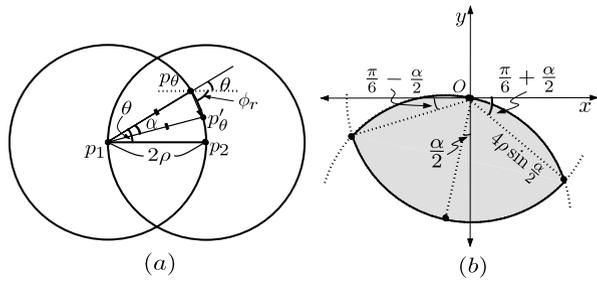
Because (p_1, p_2) is a diametrical pair, every point of P lies within a lune formed by the intersection of two discs of radius 2ρ centered at p_1 and p_2 , respectively. (See Fig. 9(a).) As before, for the sake of illustration let us assume that $\overrightarrow{p_1 p_2}$ is directed horizontally from left to right as shown in the figure, and that α is a clockwise angle. Along any fixed direction, the farther that a point is from the center of rotation p_1 the greater its rotational displacement is. So we will concentrate on the most distant points from the center of rotation, that is, points lying on the boundary of this lune. For any angle θ , where $-\pi/2 \leq \theta \leq \pi/2$, let p_θ denote the point on this lune such that the signed angle $\angle p_2 p_1 p_\theta$ is θ . (See Fig 9(a).) Let p'_θ denote this point's position after a clockwise rotation about p_1 by angle α . Let $\vec{r}(\theta) = \overrightarrow{p_\theta p'_\theta}$ be the corresponding rotational displacement vector.

The following lemma characterizes $\mathcal{R}_\rho(\alpha)$ by describing $\vec{r}(\theta)$ as a function of θ . This region is illustrated in Fig. 9(b).

Lemma 5.3 *Consider fixed α and ρ , and let*

$$s_0 = 4\rho \sin \frac{\alpha}{2}, \quad \phi_1 = -\left(\frac{5\pi}{6} + \frac{\alpha}{2}\right) \quad \text{and} \quad \phi_2 = -\left(\frac{\pi}{6} + \frac{\alpha}{2}\right).$$

Fig. 9 The rotation space for serial alignment



The rotational displacement space $\mathcal{R}_\rho(\alpha)$ is a region bounded by two circular arcs:

- (i) the arc of a circle centered at the origin of radius s_0 for $\phi \in [\phi_1, \phi_2]$.
- (ii) the arc of a circle passing through the origin whose radius is s_0 , and whose center lies at the polar coordinates $\langle s_0, -\frac{\pi+\alpha}{2} \rangle$, for $\phi \in [\phi_2, 0] \cup [-\pi, \phi_1]$.

Furthermore, for all $\vec{r} \in R_\rho(\alpha)$, we have $\|\vec{r}\| \leq s_0$.

Proof Throughout we will use the following simple observation. The triangle $\triangle p_1 p_\theta p'_\theta$ is isosceles and its apex angle is α . Thus, the length of the displacement vector $\overrightarrow{p_\theta p'_\theta}$ is the length of the triangle's base, which is $2\|p_1 p_\theta\| \sin \frac{\alpha}{2}$, and its angle with respect to $\overrightarrow{p_1 p_2}$ is $-(\frac{\pi}{2} - \theta + \frac{\alpha}{2})$. We consider two cases depending on θ , where $-\pi/2 \leq \theta \leq \pi/2$.

- (i) If $\theta \in [-\pi/3, \pi/3]$ then p_θ lies on the opposite side of the lune from p_1 , that is, on the arc of the circle of radius 2ρ centered at p_1 . Because all such points are at distance 2ρ from p_1 , by the above observation, the length of the displacement vector for all such points is $4\rho \sin \frac{\alpha}{2} = s_0$, and the direction is $\phi = -(\frac{\pi}{2} - \theta + \frac{\alpha}{2})$. For the given range of θ we have

$$\begin{aligned} \phi &\in \left[-\left(\frac{\pi}{2} + \frac{\pi}{3} + \frac{\alpha}{2}\right), -\left(\frac{\pi}{2} - \frac{\pi}{3} + \frac{\alpha}{2}\right) \right] = \left[-\left(\frac{5\pi}{6} + \frac{\alpha}{2}\right), -\left(\frac{\pi}{6} + \frac{\alpha}{2}\right) \right] \\ &= [\phi_1, \phi_2]. \end{aligned}$$

Clearly this is a circular arc centered at the origin of radius s_0 .

- (ii) If $\theta \in [\pi/3, \pi/2] \cup [-\pi/2, -\pi/3]$ then p_θ lies on the left boundary of the lune passing through p_1 . This is the arc of the circle of radius 2ρ centered at p_2 . By simple trigonometry we have

$$\|p_1 p_\theta\| = 2\rho \cdot 2 \sin \frac{\pi - 2\theta}{2} = 4\rho \cos \theta.$$

By the above observation, the length of the rotational displacement vector is $8\rho \cos \theta \sin \frac{\alpha}{2}$ and its direction is $\phi = -(\frac{\pi}{2} - \theta + \frac{\alpha}{2})$. Therefore, as a function of θ , the polar coordinates of the displacement vector are

$$\left\langle 8\rho \cos \theta \sin \frac{\alpha}{2}, -\left(\frac{\pi}{2} - \theta + \frac{\alpha}{2}\right) \right\rangle.$$

It is easy to verify algebraically the resulting curve is the arc of a circle passing through the origin whose radius is $4\rho \sin \frac{\alpha}{2} = s_0$, and whose center lies at the polar coordinates

$$\left\langle 4\rho \sin \frac{\alpha}{2}, -\frac{\pi + \alpha}{2} \right\rangle = \left\langle s_0, -\frac{\pi + \alpha}{2} \right\rangle. \quad \square$$

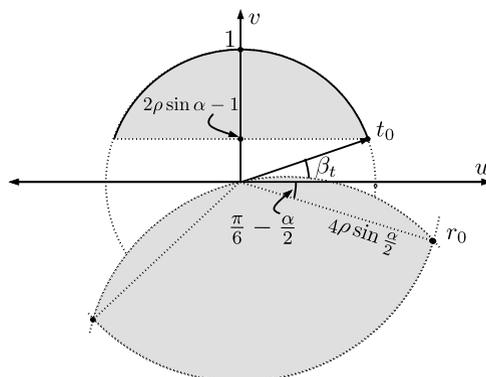
5.3 Combining Translation and Rotation

Finally we use the results of the prior two sections to derive the approximation ratio for the serial alignment algorithm. The space of possible translational displacement vectors, $\mathcal{T}_\rho(\alpha)$, was shown in Fig. 8 and space of possible rotational displacement vectors, $\mathcal{R}_\rho(\alpha)$, was shown in Fig. 9. The former was shown in (u, v) coordinates (aligned with $\overline{q_1q_2}$) and the latter was shown in (x, y) coordinates (aligned with $\overline{p_1p_2}$). We show them both in Fig. 10 in (u, v) coordinates by rotating the latter counterclockwise by angle α .

To derive the overall approximation bound it suffices to find (as a function of ρ and maximizing over all α) the two vectors $\vec{t} \in \mathcal{T}_\rho(\alpha)$ and $\vec{r} \in \mathcal{R}_\rho(\alpha)$ whose sum produces the largest total displacement distance $\|\vec{t} + \vec{r}\|$. Because each point of P is initially within distance $h_{\text{opt}} (= 1)$ of its closest point of Q , and so the final approximation bound will be $\|\vec{t} + \vec{r}\| + 1$. In order to do this, we need to determine the pair of vectors from these spaces whose sum achieves the greatest distance. A visual inspection of Fig. 10 suggests that the desired pair consists of the rightmost vertex of each region. We establish this in the following lemma.

Recall from Lemma 5.2 the angle β_t where $\sin \beta_t = 2\rho \sin \alpha - 1$, which the angle to the rightmost vertex of $\mathcal{T}_\rho(\alpha)$. Recall from the comments after Lemma 5.2 that we may assume that $\beta_t > 0$. Let \vec{t}_0 denote the unit length vector to this vertex, whose polar coordinates are $\langle 1, \beta_t \rangle$. Also recall from Lemma 5.3 that the angle (in (x, y) coordinates) to the rightmost vertex of $\mathcal{R}_\rho(\alpha)$ is $\phi_2 = -(\frac{\pi}{6} + \frac{\alpha}{2})$. To convert this into (u, v) coordinates we add α , and so let us define $\phi_r = \phi_2 + \alpha = -\frac{\pi}{6} + \frac{\alpha}{2}$. Let $\vec{r}_0 \in \mathcal{R}_\rho(\alpha)$ denote the vector to this vertex whose polar coordinates are $\langle 4\rho \sin \frac{\alpha}{2}, \phi_r \rangle$.

Fig. 10 Translation and rotation space for serial alignment



Lemma 5.4 Assuming $\beta_t > 0$,

$$\max_{\substack{\vec{t} \in \mathcal{T}_\rho(\alpha) \\ \vec{r} \in \mathcal{R}_\rho(\alpha)}} \|\vec{t} + \vec{r}\|^2 \leq 1 + \left(4\rho \sin \frac{\alpha}{2}\right)^2 + 8\rho \sin \frac{\alpha}{2} \sin\left(\frac{\pi}{3} - \beta_t + \frac{\alpha}{2}\right).$$

The sum of the vectors \vec{t}_0 and \vec{r}_0 achieves this maximum length.

Proof We will show that for all $\vec{t} \in \mathcal{T}_\rho(\alpha)$ and $\vec{r} \in \mathcal{R}_\rho(\alpha)$, $\|\vec{t} + \vec{r}\| \leq \|\vec{t}_0 + \vec{r}_0\|$. First we observe that $\phi_r < 0$. To see this recall that $\rho > \rho_{\text{ser}}^*$ and $\sin \alpha < 1/\rho$. Thus

$$\sin \alpha < \frac{1}{\rho_{\text{ser}}^*} \approx \frac{1}{1.72} < \frac{\sqrt{3}}{2} = \sin \frac{\pi}{3}.$$

Therefore $\alpha < \pi/3$, implying that $\phi_r < 0$.

Assuming that the angle between the two vectors \vec{t} and \vec{r} is acute, in order to maximize $\|\vec{t} + \vec{r}\|$ we should maximize the lengths of these vectors while minimizing the angle between them. Because the upper boundary of $\mathcal{T}_\rho(\alpha)$ is a circle centered at the origin, it is easy to see that the optimal choice would be at either the leftmost or rightmost vertices of the region. Because of the asymmetry introduced by α , the rightmost vertex t_0 is easily seen to be the better choice. (This is based on our assumption that α is a clockwise angle. The leftmost vertex would be chosen if the rotation had been counterclockwise.)

The remaining problem is to determine the best choice of \vec{r} . The lower boundary of $\mathcal{R}_\rho(\alpha)$ is a circular arc centered at the origin, and so the optimal choice cannot be in the interior of this arc. Thus, the optimal choice must lie along the upper boundary of $\mathcal{R}_\rho(\alpha)$, and in particular, along the arc running from the origin to \vec{r}_0 . We will show that the optimum is achieved at \vec{r}_0 . Recall from case (ii) of the proof of Lemma 5.3 that the points of this arc have polar coordinates (relative to (x, y) coordinates)

$$\left\langle 8\rho \cos \theta \sin \frac{\alpha}{2}, -\left(\frac{\pi}{2} - \theta + \frac{\alpha}{2}\right) \right\rangle,$$

where $\pi/3 \leq \theta \leq \pi/2$. We can convert this to polar coordinates relative to (u, v) coordinates by adding α to the angle. Let

$$\vec{r}(\theta) = \langle s(\theta), \phi(\theta) \rangle, \quad \text{where } s(\theta) = 8\rho \cos \theta \sin \frac{\alpha}{2}, \text{ and } \phi(\theta) = -\frac{\pi}{2} + \theta + \frac{\alpha}{2}.$$

Observe that $\vec{r}_0 = \vec{r}(\pi/3)$. Recall from Lemma 5.2 that the polar representation of \vec{t}_0 is $\langle 1, \beta_t \rangle$. By the law of cosines, the squared length of their sum can be expressed as a function $f(\theta) = \|\vec{t}_0 + \vec{r}(\theta)\|^2$, where

$$\begin{aligned} f(\theta) &= 1^2 + s(\theta)^2 - 2 \cdot 1 \cdot s(\theta) \cdot \cos(\pi - (\beta_t - \phi(\theta))) \\ &= 1 + \left(8\rho \cos \theta \sin \frac{\alpha}{2}\right)^2 + 16\rho \cos \theta \sin \frac{\alpha}{2} \sin\left(\theta - \beta_t + \frac{\alpha}{2}\right). \end{aligned}$$

To determine its maximum value, we consider the partial derivative of f with respect to θ .

$$\begin{aligned} \frac{\partial f}{\partial \theta} &= -128\rho^2 \sin^2 \frac{\alpha}{2} \cos \theta \sin \theta \\ &\quad + 16\rho \sin \frac{\alpha}{2} \left(-\sin \theta \sin \left(\theta - \beta_t + \frac{\alpha}{2} \right) + \cos \theta \cos \left(\theta - \beta_t + \frac{\alpha}{2} \right) \right) \\ &= -128\rho^2 \sin^2 \frac{\alpha}{2} \cos \theta \sin \theta + 16\rho \sin \frac{\alpha}{2} \cos \left(2\theta - \beta_t + \frac{\alpha}{2} \right). \end{aligned}$$

To simplify notation let $w = 8\rho \sin \frac{\alpha}{2}$, and by simple trigonometry we have

$$\begin{aligned} \frac{\partial f}{\partial \theta} &= -2w^2 \cos \theta \sin \theta + 2w \cos \left(2\theta - \beta_t + \frac{\alpha}{2} \right) \\ &= -w^2 \sin 2\theta + 2w \cos 2\theta \cos \left(\beta_t - \frac{\alpha}{2} \right) + 2w \sin 2\theta \sin \left(\beta_t - \frac{\alpha}{2} \right). \end{aligned}$$

Now we will show that the derivative is negative under our assumption that $\pi/3 \leq \theta \leq \pi/2$ and $\beta_t > 0$. Observe that $\sin 2\theta \geq 0$ and $\cos 2\theta < 0$. To estimate the other terms we first consider the range of $\beta_t - \frac{\alpha}{2}$. Using the fact that $\sin \beta_t = 2\rho \sin \alpha - 1$ and $\beta_t > 0$ it follows that $0 < \beta_t \leq \pi/2$. Since $0 \leq \alpha < \pi/3$ for $\rho > \rho_{\text{ser}}^*$ we have

$$-\frac{\pi}{6} \leq \beta_t - \frac{\alpha}{2} \leq \frac{\pi}{2}.$$

Thus, $\cos(\beta_t - \alpha/2) \geq 0$. The only remaining term is $\sin(\beta_t - \alpha/2)$. If $\beta_t - \frac{\alpha}{2} \leq 0$ it is obvious that the partial derivative is negative. Otherwise, $\beta_t - \frac{\alpha}{2} > 0$, and so

$$\begin{aligned} \frac{\partial f}{\partial \theta} &= -w^2 \sin 2\theta + 2w \cos 2\theta \cos \left(\beta_t - \frac{\alpha}{2} \right) + 2w \sin 2\theta \sin \left(\beta_t - \frac{\alpha}{2} \right) \\ &< -w^2 \sin 2\theta + 2w \sin 2\theta = (2 - w)w \sin 2\theta. \end{aligned}$$

Since $\beta_t > 0$ it follows that $2\rho \sin \alpha > 1$ and hence

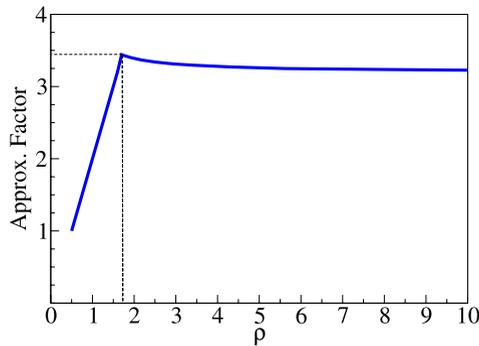
$$w = 8\rho \sin \frac{\alpha}{2} > 4\rho \sin \alpha > 2.$$

Therefore, the partial derivative is negative for $\pi/3 \leq \theta \leq \pi/2$ and $\beta_t > 0$, and so $f(\theta)$ is a decreasing function of θ . This implies that the maximum rotational displacement is achieved at the minimum value of $\theta = \frac{\pi}{3}$. Thus, the maximum combined displacement is achieved by the sum of \vec{t}_0 and $\vec{r}(\pi/3) = \vec{r}_0$, as desired. By plugging $\theta = \pi/3$ into $f(\theta)$ we obtain the maximum squared displacement distance value given in the statement of the lemma. □

Combining this result with our earlier remarks we have

$$\begin{aligned} A_{\text{ser}}(\rho) &\leq \max_{\alpha} \left(\max_{\substack{\vec{t} \in \mathcal{T}_{\rho}(\alpha) \\ \vec{r} \in \mathcal{R}_{\rho}(\alpha)}} \|\vec{t} + \vec{r}\| \right) + 1 \\ &= \max_{\alpha} \sqrt{1 + \left(4\rho \sin \frac{\alpha}{2} \right)^2 + 8\rho \sin \frac{\alpha}{2} \sin \left(\frac{\pi}{3} - \beta_t + \frac{\alpha}{2} \right)} + 1. \end{aligned}$$

Fig. 11 The approximation ratio for serial alignment as a function of ρ



Unfortunately, this function is too complex to reason about easily. However, the resulting approximation ratio can be computed numerically and plotted as a function of ρ . (See Fig 11.) By numerical means we determine that the cross-over point, denoted ρ_{ser}^* is approximately 1.72, and $A_{\text{ser}}(\rho_{\text{ser}}^*) \leq 3.44$. The figure shows that as ρ tends to ∞ the approximation bound decreases and converges to a value, which we denote by c_∞ . The next lemma provides a characterization of this function.

Lemma 5.5 *If $\rho > \rho_{\text{ser}}^*$ then $A_{\text{ser}}(\rho) \leq c_\infty + \frac{9}{4\rho}$, where $c_\infty = \lim_{\rho \rightarrow \infty} A_{\text{ser}}(\rho) \approx 3.196$.*

Proof We will follow the approach we used in Lemma 3.5. We first find an optimal solution for the limiting case when $\rho \rightarrow \infty$ and then present the general case. Let $D_\rho(\alpha)$ denote the maximum squared total displacement length as given in Lemma 5.4. When ρ and α are clear from context we simply write D .

$$D = D_\rho(\alpha) = 1 + \left(4\rho \sin \frac{\alpha}{2}\right)^2 + 8\rho \sin \frac{\alpha}{2} \sin\left(\frac{\pi}{3} - \beta_t + \frac{\alpha}{2}\right).$$

As mentioned above, $A_{\text{ser}}(\rho) \leq \max_\alpha \sqrt{D} + 1$. Since $0 \leq \sin \alpha \leq 1/\rho$, in the limit α approaches 0. Using the fact of $\lim_{\alpha \rightarrow 0} \frac{\sin \alpha}{\alpha} = 1$ and elementary trigonometry we have

$$\begin{aligned} \lim_{\rho \rightarrow \infty} D &= \lim_{\rho \rightarrow \infty} \left(1 + \left(4\rho \sin \frac{\alpha}{2}\right)^2 + 8\rho \sin \frac{\alpha}{2} \left(\sin\left(\frac{\pi}{3} + \frac{\alpha}{2}\right) \cos \beta_t - \sin \beta_t \cos\left(\frac{\pi}{3} + \frac{\alpha}{2}\right)\right)\right) \\ &= \lim_{\rho \rightarrow \infty} 1 + (2\rho\alpha)^2 + 4\rho\alpha \left(\frac{\sqrt{3}}{2} \sqrt{4\rho\alpha - 4\rho^2\alpha^2} - (2\rho\alpha - 1)\frac{1}{2}\right) \\ &= \lim_{\rho \rightarrow \infty} 1 + 2\rho\alpha + 4\sqrt{3}\rho\alpha \sqrt{\rho\alpha - \rho^2\alpha^2}. \end{aligned}$$

Substituting $x = \rho\alpha(0 \leq x \leq 1)$, let

$$f(x) = 1 + 2x + 4\sqrt{3}x\sqrt{x - x^2}.$$

To find the value $x = x_0$ at which the maximum is achieved, we set the derivative to 0, from which we obtain

$$x_0 = \frac{1}{6} \left(3 + \sqrt{2} \cos \phi_0 + \sqrt{6} \sin \phi_0 \right), \quad \text{where } \phi_0 = \frac{1}{3} \arctan \frac{\sqrt{47}}{9}. \quad (2)$$

Observe that x_0 is slightly greater than $\sqrt{2/3}$. Thus, we have

$$c_\infty = A_{\text{ser}}(\infty) = \lim_{\rho \rightarrow \infty} \max_{\alpha} \sqrt{D_\rho(\alpha)} + 1 = \sqrt{f(x_0)} + 1 \approx 3.196. \quad (3)$$

Next, we consider the general case. Before considering D , let us simplify the last term of D first. Let

$$g_\rho(\alpha) = \sin \left(\frac{\pi}{3} - \beta_t + \frac{\alpha}{2} \right).$$

From basic trigonometry we have

$$\begin{aligned} g_\rho(\alpha) &= \sin \left(\frac{\pi}{3} + \frac{\alpha}{2} \right) \cos \beta_t - \sin \beta_t \cos \left(\frac{\pi}{3} + \frac{\alpha}{2} \right) \\ &= \sin \frac{\pi}{3} \cos \frac{\alpha}{2} \cos \beta_t + \cos \frac{\pi}{3} \sin \frac{\alpha}{2} \cos \beta_t - \cos \frac{\pi}{3} \cos \frac{\alpha}{2} \sin \beta_t \\ &\quad + \sin \frac{\pi}{3} \sin \frac{\alpha}{2} \sin \beta_t. \end{aligned}$$

To simplify $g_\rho(\alpha)$ we apply a Taylor’s series expansion.

$$\begin{aligned} g_\rho(\alpha) &\leq \frac{\sqrt{3}}{2} \cos \beta_t + \frac{\alpha}{4} \cos \beta_t - \frac{1}{2} \left(1 - \frac{\alpha^2}{8} \right) \sin \beta_t + \frac{\sqrt{3}\alpha}{4} \sin \beta_t \\ &\leq \left(\sqrt{3} + \frac{\alpha}{2} \right) \sqrt{\rho\alpha - \rho^2 \left(\alpha - \frac{\alpha^3}{6} \right)^2} \\ &\quad - \frac{1}{2} \left(1 - \frac{\alpha^2}{8} \right) \left(2\rho \left(\alpha - \frac{\alpha^3}{6} \right) - 1 \right) + \frac{\sqrt{3}\alpha}{4} (2\rho\alpha - 1). \end{aligned}$$

Now, applying a Taylor’s series expansion of D (as a function of α) we have

$$\begin{aligned} D &= 1 + \left(4\rho \sin \frac{\alpha}{2} \right)^2 + 8\rho \sin \frac{\alpha}{2} \cdot g_\rho(\alpha) \\ &\leq 1 + (2\rho\alpha)^2 + 4\rho\alpha \cdot g_\rho(\alpha). \end{aligned}$$

Substituting $g_\rho(\alpha)$ for the last term of D and removing all the negative terms we have

$$D \leq 1 + 2\rho\alpha + 4\rho\alpha \left(\left(\sqrt{3} + \frac{\alpha}{2} \right) \sqrt{\rho\alpha - \rho^2\alpha^2 + \frac{\rho^2\alpha^4}{3} + \frac{7\rho\alpha^3}{24} + \frac{\sqrt{3}\rho\alpha^2}{2}} \right). \quad (4)$$

Since $0 \leq \alpha \leq \arcsin(1/\rho)$, we have $0 \leq \rho\alpha \leq \rho \arcsin(1/\rho)$. For all $\rho > \rho_{\text{ser}}^*$, observe that $\rho \arcsin(1/\rho)$ is at most $\rho_{\text{ser}}^* \arcsin(1/\rho_{\text{ser}}^*)$. Let us denote this value by c_0 , which by numerical evaluation is approximately 1.07. Thus, we have $0 \leq \rho\alpha < c_0$.

We will show that $D \leq (c_\infty - 1)^2 + \frac{9}{\rho}$. We will consider two cases: $0 \leq \rho\alpha \leq 1$ and $1 < \rho\alpha < c_0$. For the first case ($0 \leq \rho\alpha \leq 1$) we substitute $x = \rho\alpha$ ($0 \leq x \leq 1$) and use the fact that $(1 + 2x + 4\sqrt{3}x\sqrt{x - x^2}) \leq f(x_0)$, which we argued above. And so we have

$$\begin{aligned} D &\leq D + \left[f(x_0) - \left(1 + 2x + 4\sqrt{3}x\sqrt{x - x^2} \right) \right] \\ &= f(x_0) + 4\sqrt{3}x \left(\sqrt{x - x^2 + \frac{x^4}{3\rho^2}} - \sqrt{x - x^2} \right) \\ &\quad + 4x \left(\frac{x}{2\rho} \sqrt{x - x^2 + \frac{x^4}{3\rho^2}} + \frac{7x^3}{24\rho^2} + \frac{\sqrt{3}x^2}{2\rho} \right). \end{aligned}$$

It is easily proved that this function is increasing for $0 \leq x \leq 1$. Thus, it achieves its maximum value at $x = 1$. By hypothesis, $\rho > \rho_{\text{ser}}^* > 1.72$, from which it follows that

$$D \leq f(x_0) + \left(\frac{4 + 2\sqrt{3}}{\rho} + \frac{7 + 4\sqrt{3}}{6\rho^2} \right) < f(x_0) + \frac{9}{\rho} = (c_\infty - 1)^2 + \frac{9}{\rho},$$

as desired.

For the second case ($1 < \rho\alpha < c_0$), since $\rho\alpha - \rho^2\alpha^2 < 0$, from (4) we have

$$D < 1 + 2\rho\alpha + 4\rho\alpha \left(\left(\sqrt{3} + \frac{\alpha}{2} \right) \sqrt{\frac{\rho^2\alpha^4}{3} + \frac{7\rho\alpha^3}{24} + \frac{\sqrt{3}\rho\alpha^2}{2}} \right).$$

It is easily observed that the function is an increasing function with respect to $\rho\alpha$ and the maximum value is achieved at $\rho\alpha = 1.07$, and this value is at most $(c_\infty - 1)^2 + \frac{9}{\rho}$, as desired.

Now, using the fact of $c_\infty > 3$ we have

$$D < (c_\infty - 1)^2 + \frac{9}{\rho} < \left(c_\infty - 1 + \frac{9}{4\rho} \right)^2.$$

Therefore,

$$A_{\text{ser}}(\rho) \leq \max_{\alpha} \sqrt{D} + 1 \leq c_\infty + \frac{9}{4\rho},$$

which completes the proof. □

The analysis of Lemma 5.4 was performed under the assumption that $\beta_t > 0$. We can justify this assumption now by showing that any other value would lead to a weaker approximation bound.

Lemma 5.6 For $\rho > \rho_{\text{ser}}^*$, if $\beta_t \leq 0$ then

$$d_{\text{ser}}(\rho) < A_{\text{ser}}(\rho)h_{\text{opt}},$$

where $d_{\text{ser}}(\rho)$ is a Hausdorff distance after running serial alignment.

Proof To simplify matters, assume that $h_{\text{opt}} = 1$ and let $d_{\text{ser}}(\rho) \leq \max_{\alpha} g_{\rho}(\alpha)$. By the triangle inequality we have

$$g_{\rho}(\alpha) = \max_{\substack{\vec{i} \in \mathcal{T}_{\rho}(\alpha) \\ \vec{r} \in \mathcal{R}_{\rho}(\alpha)}} (\|\vec{i} + \vec{r}\| + 1) \leq \max_{\substack{\vec{i} \in \mathcal{T}_{\rho}(\alpha) \\ \vec{r} \in \mathcal{R}_{\rho}(\alpha)}} (\|\vec{i}\| + \|\vec{r}\| + 1).$$

By Lemmas 5.2 and 5.3 we have $\|\vec{i}\| \leq 1$ and $\|\vec{r}\| \leq 4\rho \sin \frac{\alpha}{2}$, and so

$$g_{\rho}(\alpha) \leq 2 + 4\rho \sin \frac{\alpha}{2} = 2 + \frac{2\rho \sin \alpha}{\cos \frac{\alpha}{2}}.$$

By the assumption that $\beta_t \leq 0$, it follows $2\rho \sin \alpha - 1 \leq 0$, and so $g_{\rho}(\alpha) \leq 2 + \frac{1}{\cos(\alpha/2)}$. For $\rho > \rho_{\text{ser}}^*$ and $\sin \alpha \leq 1/(2\rho)$, we have

$$\cos \frac{\alpha}{2} = \frac{1}{\sqrt{2}} \sqrt{1 + \sqrt{1 - \sin^2 \alpha}} > \frac{1}{\sqrt{2}} \sqrt{1 + \sqrt{1 - 1/(2\rho_{\text{ser}}^*)^2}} \approx \frac{1}{1.01}.$$

Thus, we see that the bound on $d_{\text{ser}}(\rho)$ generated under the hypothesis $\beta_t \leq 0$ is significantly smaller than c_{∞} . In the proof of Lemma 5.5 it was shown that for all $\rho > \rho_{\text{ser}}^*$, the approximation bound is never smaller than c_{∞} . □

Combining Lemmas 5.1, 5.5, and 5.6 completes the proof of Theorem 3.

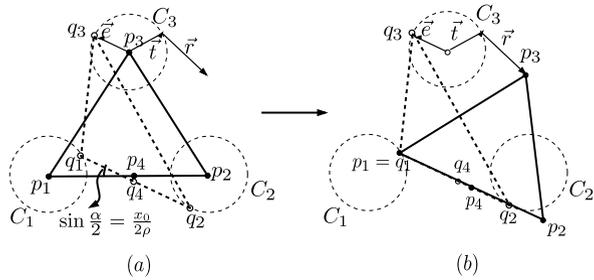
6 Serial Alignment: Lower Bound

In this section we present a result analogous to Theorem 2, by proving that the upper bound on the approximation ratio for serial alignment given in the previous section is tight in the limit as ρ approaches ∞ . Recall that $c_{\infty} = \lim_{\rho \rightarrow \infty} A_{\text{ser}}(\rho)$. (See (3) in Lemma 5.5.)

Theorem 4 For all sufficiently large ρ there exists an input on which serial alignment achieves an approximation factor of at least $c_{\infty} + \frac{1}{27\rho^2}$.

The input configuration is structurally similar to the one used in the proof of Theorem 2, except for the choice of the angle α . Here we select α such that $\sin \frac{\alpha}{2} = x_0/(2\rho)$, where x_0 is as defined in (2) in the proof of Lemma 5.5. Consider a fixed value $\rho > 2A_{\text{ser}}$. As in Sect. 4 we define the pattern point set $P = \{p_1, p_2, p_3, p_4\}$, where the first three points form an equilateral triangle of side length 2ρ , and p_4 is placed at the midpoint of $\overline{p_1 p_2}$. (See Fig. 12(a)) By an infinitesimal perturbation of the points, we may assume that the pair (p_1, p_2) is the unique diametrical pair for P .

Fig. 12 The lower bound on A_{ser}



We define the background set $Q = \{q_1, q_2, q_3, q_4\}$ so that the optimum Hausdorff distance will be at most 1. For $1 \leq i \leq 4$, let C_i be a circle of unit radius centered p_i . Consider a lower tangent line at C_2 forming an angle α with line $\overline{p_1 p_2}$. It is easy to verify that this line intersects C_1 . Place q_1 at the rightmost intersection point of the tangent line and C_1 , and place q_2 at the point of tangency with C_2 . (See Fig. 12(a).) Running serial alignment so that (p_1, p_2) is aligned with (q_1, q_2) results in a translation and rotation. Let \vec{t} and \vec{r} denote the translation and rotational displacement vectors for p_3 , respectively. Let $q_3 = p_3 + \vec{e}$, where \vec{e} is a vector of unit length whose direction is chosen to be directly opposite that of $\vec{t} + \vec{r}$. Intuitively, q_3 has been chosen to be as far away as possible from p_3 after alignment. Finally, place q_4 at the midpoint of $\overline{q_1 q_2}$. Observe that prior to alignment, each p_i is within distance 1 from q_i , and therefore $h_{opt} \leq 1$.

Consider the placement of P and Q after running serial alignment. (See Fig. 12(b).) We will analyze the displacement distance of p_3 relative to q_3 . As in the construction of Sect. 4, p_4 and q_4 were introduced to induce an asymmetry that forces the alignment of (p_1, p_2) with (q_1, q_2) . We establish this formally in the next lemma.

Lemma 6.1 *For all sufficiently large ρ , serial alignment will align (p_1, p_2) with (q_1, q_2) .*

Proof We reduce the proof to the analogous result for symmetric alignment, namely Lemma 4.1. In the proof of that lemma the only facts that we used were that ρ is sufficiently large, and prior to alignment, each point of P lies within distance $h_{opt} \leq 1$ of the corresponding point of Q . These are both true in the present scenario.

Suppose that when we run serial alignment on the above input instance (p_1, p_2) was not aligned with (q_1, q_2) . In Lemma 2.3 we showed that on all input instances, if serial alignment produces a Hausdorff distance of d_{ser} then (without altering the individual point assignments) symmetric alignment will produce a Hausdorff distance of at most $d_{sym} \leq d_{ser} + h_{opt}$. In the proof of Lemma 4.1 it was argued that if (p_1, p_2) was not aligned with (q_1, q_2) in symmetric alignment, the resulting Hausdorff distance would be at least $\rho - h_{opt}$. Thus we have

$$\rho - h_{opt} \leq d_{sym} \leq d_{ser} + h_{opt},$$

which implies that $d_{ser} \geq \rho - 2h_{opt} \geq \rho - 2$. This distance will exceed $A_{ser}h_{opt}$ for all sufficiently large ρ , which yields the desired contradiction. \square

To complete the analysis let us consider the translational and rotational displacement distances of p_3 that result from symmetric alignment. It is easy to see that $\vec{t} = \overline{p_1 q_1}$, and so $\|\vec{t}\| = 1$. To determine the direction of \vec{t} , let us consider the points of P prior to alignment. Let ℓ denote the line passing through p_1 that is parallel to $\overline{q_1 q_2}$. Let h_1 and h_2 denote the orthogonal projections of q_1 and q_2 onto ℓ , respectively. Since line $\overline{q_1 q_2}$ is the lower tangent line of the circle C_2 at q_2 , it follows that p_2, q_2 , and h_2 are collinear. Combining the facts that $\|p_1 p_2\| = 2\rho$, $\|p_2 q_2\| = 1$, and ℓ is perpendicular to $\overline{p_2 q_2}$, we have

$$\|q_2 h_2\| = \|p_1 p_2\| \sin \alpha - \|p_2 q_2\| = 2\rho \sin \alpha - 1.$$

Because $\|q_1 h_1\|$ is equal to $\|q_2 h_2\|$, the angle between the line $\overline{p_1 q_1}$ and ℓ , denoted β_t , satisfies

$$\sin \beta_t = \frac{\|q_1 h_1\|}{\|p_1 q_1\|} = \frac{\|q_2 h_2\|}{1} = 2\rho \sin \alpha - 1.$$

Since ℓ forms an angle of α with $\overline{p_1 p_2}$ it follows that \vec{t} forms an angle of $\phi_t = \beta_t - \alpha$, and hence expressing \vec{t} in polar coordinates we have $\vec{t} = \langle 1, \phi_t \rangle$.

Next, let us consider the effect of rotation at p_3 . Since $\Delta p_1 p_2 p_3$ is an equilateral triangle we have $\|p_1 p_3\| = 2\rho$. By simple trigonometry (or see the comments at the start of the proof of Lemma 5.3) it follows that p_3 's displacement distance due to the rotation about p_1 by angle α is $2\rho \cdot 2 \sin \frac{\alpha}{2} = 4\rho \sin \frac{\alpha}{2}$ and the displacement direction is

$$\left(\frac{\pi}{3} - \frac{\alpha}{2}\right) - \frac{\pi}{2} = -\frac{\pi}{6} - \frac{\alpha}{2}.$$

Let ϕ_r denote this angle. Thus, expressing \vec{r} in polar coordinates we have

$$\vec{r} = \left\langle 4\rho \sin \frac{\alpha}{2}, \phi_r \right\rangle.$$

Let $D = \|\vec{t} + \vec{r}\|^2$ denote the length of the combined displacements. By the law of cosines we have

$$\begin{aligned} D &= \|\vec{t}\|^2 + \|\vec{r}\|^2 - 2\|\vec{t}\| \|\vec{r}\| \cos(\pi - (\phi_t - \phi_r)) \\ &= 1 + \left(4\rho \sin \frac{\alpha}{2}\right)^2 - 8\rho \sin \frac{\alpha}{2} \cos\left(\pi - (\beta_t - \alpha) + \left(-\frac{\pi}{6} - \frac{\alpha}{2}\right)\right) \\ &= 1 + \left(4\rho \sin \frac{\alpha}{2}\right)^2 + 8\rho \sin \frac{\alpha}{2} \sin\left(\frac{\pi}{3} + \frac{\alpha}{2} - \beta_t\right). \end{aligned}$$

Using the facts that $\sin \frac{\alpha}{2} = \frac{x_0}{2\rho}$ and $0 \leq \alpha \leq \pi/3$ and applying basic trigonometry we have

$$\begin{aligned} D &= 1 + (2x_0)^2 + 4x_0 \left(\sin\left(\frac{\pi}{3} + \frac{\alpha}{2}\right) \cos \beta_t - \cos\left(\frac{\pi}{3} + \frac{\alpha}{2}\right) \sin \beta_t\right) \\ &\geq 1 + 4x_0^2 + 4x_0 \left(\frac{\sqrt{3}}{2} \cos \beta_t - \frac{1}{2} \sin \beta_t\right). \end{aligned}$$

Note that x_0 is a constant slightly greater than $\sqrt{2/3}$. (See (2) in Lemma. 5.5.) To relate this to c_∞ we begin with a variable substitution. Let

$$s = \cos \frac{\alpha}{2} = \sqrt{1 - \left(\frac{x_0}{2\rho}\right)^2}.$$

Because $0 \leq x_0 < 1$ and $\rho > \rho_{\text{ser}}^* > 1.5$ it follows directly that $3/4 < s \leq 1$. We can restate a number of quantities in terms of x_0 and s .

$$\begin{aligned} \rho \sin \alpha &= 2\rho \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = x_0 s, \\ \sin \beta_t &= 2\rho \sin \alpha - 1 = 2x_0 s - 1, \quad \text{and} \\ \cos \beta_t &= 2\sqrt{x_0 s - x_0^2 s^2}. \end{aligned}$$

Substituting these values yields

$$D \geq 1 + 4x_0^2 + 4x_0 \left(\sqrt{3}\sqrt{x_0 s - x_0^2 s^2} - x_0 s + \frac{1}{2} \right).$$

By definition of c_∞ we have $(c_\infty - 1)^2 = 1 + 2x_0 + 4\sqrt{3}x_0\sqrt{x_0 - x_0^2}$, and so it follows that

$$\begin{aligned} D &\geq 1 + 4x_0^2 + 4x_0 \left(\sqrt{3}\sqrt{x_0 s - x_0^2 s^2} - x_0 s + \frac{1}{2} \right) \\ &\quad + \left[(c_\infty - 1)^2 - \left(1 + 2x_0 + 4\sqrt{3}x_0\sqrt{x_0 - x_0^2} \right) \right] \\ &= (c_\infty - 1)^2 + 4x_0^2(1 - s) + 4\sqrt{3}x_0 \left(\sqrt{x_0 s - x_0^2 s^2} - \sqrt{x_0 - x_0^2} \right). \end{aligned}$$

It is straightforward to verify that $1/(2x_0) < s \leq 1$ for $\rho > \rho_{\text{ser}}^*$, and so it follows that $\sqrt{x_0 s - x_0^2 s^2} - \sqrt{x_0 - x_0^2} \geq 0$. We have

$$D \geq (c_\infty - 1)^2 + 4x_0^2(1 - s).$$

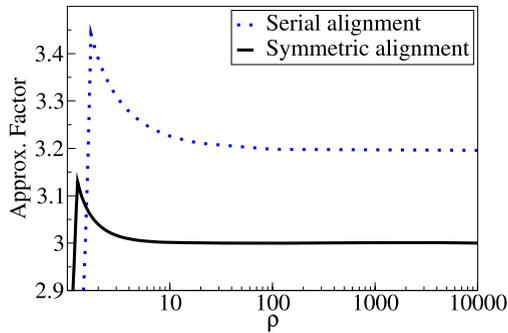
Using the fact that $\sqrt{1 - x} \leq 1 - x/2$ and $s = \sqrt{1 - \left(\frac{x_0}{2\rho}\right)^2}$ we have

$$D \geq (c_\infty - 1)^2 + 4x_0^2 \left[1 - \left(1 - \frac{1}{2} \left(\frac{x_0}{2\rho} \right)^2 \right) \right] = (c_\infty - 1)^2 + \frac{x_0^4}{2\rho^2}.$$

Since $c_\infty < 3.2$ and $\sqrt{2/3} < x_0 < 1$ it is easy to verify that

$$D \geq (c_\infty - 1)^2 + \frac{x_0^4}{2\rho^2} \geq \left(c_\infty - 1 + \frac{x_0^4}{12\rho^2} \right)^2 \geq \left(c_\infty - 1 + \frac{1}{27\rho^2} \right)^2.$$

Fig. 13 The approximation ratios for serial alignment and symmetric alignment (Note that the y-axis does not start at 0)



Therefore,

$$A_{\text{ser}}(\rho) \geq \frac{1 + \sqrt{D}}{h_{\text{opt}}} \geq c_{\infty} + \frac{1}{27\rho^2},$$

which completes the proof of Theorem 4.

7 Summary and Concluding Remarks

We have presented a simple modification to the alignment-based algorithm of Goodrich, Mitchell, and Orletsky [13]. Our modification has the same running time and retains the simplicity of the original algorithm. We have analyzed the approximation ratios for these algorithms as a function of the distance ratio ρ . We summarize and compare these approximation ratios in Fig. 13.

Note that the approximation ratio for symmetric alignment is smaller for all sufficiently large values of ρ . The cross-over point of two bounds is at $(\rho, A) \approx (1.54, 3.08)$. As a function of ρ , the highest upper bound for symmetric alignment is at $(\rho, A_{\text{sym}}) \approx (1.26, 3.14)$. The highest upper bound for serial alignment is at $(\rho, A_{\text{ser}}) \approx (1.72, 3.44)$. As ρ approaches ∞ , $A_{\text{sym}}(\rho)$ converges to 3 and $A_{\text{ser}}(\rho)$ converges to a value that is approximately 3.19. As mentioned earlier, in many applications of point pattern matching, large values of ρ (exceeding 10, say) are to be expected. We have also shown that both bounds are nearly tight for large ρ , and they are tight in the limit.

This paper opens the question of what are the performance limits of approximations based on point alignments. It is natural to consider generalizations of this approach to higher dimensions, to other geometric groups (including, for example uniform scaling), to matchings based on more than two points, and to other sorts of point-set similarity measures.

We would like to thank the reviewers for their insightful comments, which improved the quality of the paper.

References

1. Alt, H., Aichholzer, O., Rote, G.: Matching shapes with a reference point. In: Proceedings of the 10th Annual ACM Symposium on Comput. Geom., pp. 85–92 (1994)

2. Alt, H., Guibas, L.J.: Discrete geometric shapes: matching, interpolation, and approximation. Technical report 96–11, Institut für Informatik, Freie Universität Berlin, Berlin, Germany (1996)
3. Alt, H., Mehlhorn, K., Wagener, H., Welzl, E.: Congruence, similarity and symmetries of geometric objects. *Discrete Comput. Geom.* **3**, 237–256 (1988)
4. Alzina, M., Szpankowski, W., Grama, A.: 2d-pattern matching image and video compression: theory, algorithms, and experiments. *IEEE Trans. Image Process.* **11**, 318–331 (2002)
5. Cardoze, D.E., Schulman, L.: Pattern matching for spatial point sets. In: Proceedings of the 39th Annual IEEE Symposium on Found. Comput. Sci., pp. 156–165 (1998)
6. Chew, L.P., Goodrich, M.T., Huttenlocher, D.P., Kedem, K., Kleinberg, J.M., Kravets, D.: Geometric pattern matching under Euclidean motion. *Comput. Geom. Theory Appl.* **7**, 113–124 (1997)
7. de Berg, M., van Kreveld, M., Overmars, M.H., Schwarzkopf, O.: *Computational Geometry: Algorithms and Applications*, 2nd edn. Springer, Berlin (2000)
8. de Rezende, P.J., Lee, D.T.: Point set pattern matching in d -dimensions. *Algorithmica* **13**, 387–404 (1995)
9. Efrat, A., Itai, A.: Improvements on bottleneck matching and related problems using geometry. In: Proceedings of the 12th Annual ACM Symposium on Comput. Geom., pp. 301–310, Philadelphia, 1996
10. Finn, P., Kavraki, L.E., Latombe, J.C., Motwani, R., Shelton, C., Venkatasubramanian, S., Yao, A.: Rapid: Randomized pharmacophore identification for drug design. In: Proceedings of the 13th Annual ACM Symposium on Comput. Geom., pp. 324–333 (1997)
11. Fischler, M.A., Bolles, R.C.: Random sample consensus: a paradigm for model fitting with applications to image analysis and automated cartography. *Commun. ACM* **24**, 381–395 (1981)
12. Gavrilov, M., Indyk, P., Motwani, R., Venkatasubramanian, S.: Geometric pattern matching: a performance study. In: Proceedings of the 15th Annual ACM Symposium on Comput. Geom., pp. 79–85 (1999)
13. Goodrich, M.T., Mitchell, J.S.B., Orletsky, M.W.: Approximate geometric pattern matching under rigid motions. *IEEE Trans. Pattern Anal. Mach. Intell.* **21**(4), 371–379 (1999)
14. Heffernan, P.J., Schirra, S.: Approximate decision algorithms for point set congruence. *Comput. Geom. Theory Appl.* **4**, 137–156 (1994)
15. Hopcroft, J.E., Huttenlocher, D.P., Wayner, P.C.: *Affine Invariants for Model-Based Recognition*. MIT Press, Cambridge (1992)
16. Huttenlocher, D.P., Kedem, K., Sharir, M.: The upper envelope of Voronoi surfaces and its applications. *Discrete Comput. Geom.* **9**, 267–291 (1993)
17. Indyk, P., Motwani, R., Venkatasubramanian, S.: Geometric matching under noise: combinatorial bounds and algorithms. In: Proceedings of the 10th Annual ACM–SIAM Symposium on Discrete Algorithms, pp. 457–465 (1999)
18. Indyk, P., Venkatasubramanian, S.: Approximate congruence in nearly linear time. In: Proceedings of the 11th Annual ACM–SIAM Symposium on Discrete Algorithms, pp. 354–360, San Francisco, 2000
19. Mount, D.M., Netanyahu, N.S., Le Moigne, J.: Efficient algorithms for robust point pattern matching. *Pattern Recognit.* **32**, 17–38 (1999)
20. van Kreveld, M.J., de Berg, M.T.: Finding squares and rectangles in sets of points. *BIT* **31**, 202–219 (1991)
21. Veltkamp, R.C.: Shape matching: similarity measures and algorithms. Technical report UU-CS 2001-033, Utrecht University: Information and Computing Sciences, Utrecht, The Netherlands (2001)