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A Formal Model of Diagnostic Inference. II. Algorithmic Solution and Application

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ABSTRACT

This paper and a preceding companion paper present the generalized set-covering (GSC) formalization of diagnostic inference. In the current paper, the GSC model is used as the basis for algorithms modeling the "hypothesize-and-test" nature of diagnostic problem solving. Two situations are addressed: "concurrent" problem solving, in which all occurring manifestations are already known, and sequential problem solving, in which the manifestations are discovered one at a time. Each algorithm is explained and its correctness within the GSC framework is proven. The utility of the GSC model is illustrated by using it to describe and analyze some recent abductive expert systems for diagnostic problem solving. The limitations of the basic form of the GSC model are then discussed. A more general notion of "parsimonious covering" that includes the GSC model as a special case is then identified, and some important directions for further research are presented.

INTRODUCTION

In Part I [41], a formalization of diagnostic problems and their solution was presented. The resulting model, called the GSC model, is based on a generalization of the set-covering problem. It extends the "classical" problem in two ways:

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by permitting the covering to be inexact, and by finding all minimal covers rather than a single one. This latter extension involved introducing the notion of "generator sets" for the solution of a diagnostic problem and operations on them.

To complete the GSC model of diagnostic inference it is necessary to go one step further and model the underlying problem-solving process that is involved. This is done in the current paper by specifying algorithms that use the GSC model to find solutions for diagnostic problems. These algorithms model the hypothesize-and-test nature of human diagnostic reasoning, and have been used to build functioning expert systems.

As noted in Part I, empirical studies have shown that diagnostic reasoning involves a sequential hypothesize-and-test (abductive) process during which the diagnostician conceptually constructs a model of what is wrong. This model, referred to as the hypothesis, is based largely on what manifestations are known to be present. The hypothesis postulates the presence of one or more disorders that could explain these manifestations, and is constructed in a sequential fashion during problem solving. For example, on seeing a patient a physician does not initially know all of the manifestations which are present. Rather, given information about a few manifestations, the physician first constructs an initial hypothesis. Then, based at least in part on this hypothesis, the physician generates "questions" that test its validity (hence the term "hypothesize-and-test"). These questions uncover the presence or absence of additional manifestations, thus possibly introducing new disorders into the hypothesis, confirming or eliminating disorders already suspected to be present, or discriminating among disorders that are alternatives to one another.

The algorithms presented in the current paper provide a formal model of this sequential hypothesize-and-test process. First, it is shown how to derive the solution to a "concurrent" diagnostic problem where all of the manifestations are known initially. Next, a more realistic sequential diagnostic approach is developed where the manifestations which are present are not all known initially but must be discovered in a sequential fashion. After each algorithm is described, a proof of its correctness is given. The sequential algorithm is then compared with some existing AI programs claiming to model diagnostic reasoning. Finally, the limitations of the GSC model are discussed and a more general notion of "parsimonious covering" is introduced. Directions where further research is needed are also presented. As in Part I, an appendix contains the proofs of all assertions made in this paper.

CONCURRENT PROBLEM SOLVING

In "concurrent" problem solving one is given a diagnostic problem $\mathbf{P} = \langle \mathbf{D}, \mathbf{M}, \mathbf{C}, M^+ \rangle$, where M^+ is completely known. The goal is to derive a

```
function Solve[P]
     variables n  integer, s generator-set;
     begin
        n:=0; s:=Ø;      (* initialize n ≤ order(P); s = Ø indicates solution unknown *)
        while s=Ø  do
           begin
              s: = Genset[causes(M+), M+,n];
                                              (* s is assigned Ø if n < order(P), generator
                                                          set for Sol(P) if n = order(P) *)
              n:= n+1                                       (* increment n and try again *)
           end;
        return  s                                   (* return generator set for Sol(P) *)
     end.
```

```
function Genset[scope,manifs,n]
     variables  I set-of-disorders, F G H generator-set;
     if  n=0
        then
           if  manifs = Ø (* check if order(R) = 0, where R = prob(scope,manifs) *)
              then return  {Ø}       (* generator set for Sol(R) where order(R)=0 *)
              else return   Ø                               (* n < order(R) so fail *)
           endif
        else
           if  |scope| < n
              then return  Ø             (* no explanations of size n possible in R *)
              else                          (* recursively try to construct Sol(R) *)
                 select d ε scope,
                 I:= {d' ε scope|man(d') ∧ manifs = man(d) ∧ manifs }
                 F:= Genset[scope-I,manifs,n],
                 H:= Genset[scope-I,manifs-man(d),n-1];
                 G:= {H_i • (I)|H_i ε H};
                 return  G v F               (* return Ø if n < order(R) or generator
                                                          set for Sol(R) if n = order(R) *)
           endif
     endif.
```

Fig. 1. Formalized algorithms for solving concurrent diagnostic problems. Reserved words are in bold type. Comments are enclosed in (*...*).

generator set for Sol(P). While real-world diagnostic problems are generally more sequential in nature, development of a concurrent problem-solving algorithm is useful in that it is conceptually simpler and it introduces concepts and procedures that will be used during sequential problem solving.

Figure 1 presents two formalized algorithms, Solve and Genset, which solve concurrent diagnostic problems. The real work during problem solving is done by the recursive function Genset, which takes three input parameters:

scope: a set of disorders,
manifs: a subset of M^+, and
$n \geq 0$: a nonnegative integer.

Informally, Genset[scope, manifs, n] attempts to find a generator set which represents all explanations (if any) in scope of size n for manifs. If n is less than the actual size of an explanation for manifs, Genset returns Ø indicating failure. If n is equal to the actual size of an explanation for manifs, Genset will succeed.

Genset is not permitted to be called with n greater than the size of an explanation for manifs. The function Solve takes a diagnostic problem P and starts with $n=0$, calling Genset with scope = causes(M^+) and manifs = M^+. The value of n is subsequently incremented in Solve until Genset succeeds and returns Sol(P) in the form of a generator set.

The heart of Genset is the second nested **if** statement. The **else** part of this statement (lines 13–18) first finds a set I of disorders which all cover the same subset of manifs. Two recursive calls are then made to *Genset*. The first of these calls (line 15) finds the generator set F for all explanations in scope for manifs of size n which do not include I. The second of these calls (line 16) finds the generator set H for the "reduced" problem having manifestations manifs-man(I) and explanations of size $n-1$. Generators in H are then composed with the generator (I) to form G (line 17). G will be shown later to be the generator set for all explanations in scope for manifs of size n which do include I. Hence, $G \cup F$ (line 18) is the desired result.

In establishing the correctness of functions Solve and Genset, we will frequently discuss more than one diagnostic problem at a time. In such situations it often proves useful to use the problem names as subscripts to avoid ambiguity. Thus, the components of diagnostic problems P and Q would be identified as $P = \langle \mathbf{D}_P, \mathbf{M}_P, \mathbf{C}_P, M_P^+ \rangle$ and $Q = \langle \mathbf{D}_Q, \mathbf{M}_Q, \mathbf{C}_Q, M_Q^+ \rangle$. Similarly, the abbreviation $\text{man}_P(D)$ will be used for "man(D) in P" where $D \subseteq \mathbf{D}_P$, $\text{causes}_Q(M)$ will be used for "causes(M) in Q" where $M \subseteq \mathbf{M}_Q$, etc. In situations where the meaning is clear from context we will omit such subscripts (e.g., in Figure 1, line 14, we refer to $\text{man}_P(d)$ without a subscript, since P is the only diagnostic problem in the figure).

The following lemmas and corollaries will be used in the proofs of correctness for functions Solve and Genset.

LEMMA 2.1. *Let* $P = \langle \mathbf{D}_P, \mathbf{M}_P, \mathbf{C}_P, M_P^+ \rangle$ *and* $Q = \langle \mathbf{D}_Q, \mathbf{M}_Q, \mathbf{C}_Q, M_Q^+ \rangle$ *be diagnostic problems, and let* $D \subseteq \mathbf{D}_P \cap \mathbf{D}_Q$ *and* $M \subseteq \mathbf{M}_P \cap \mathbf{M}_Q$. *If* $\mathbf{C}_Q \subseteq \mathbf{C}_P$, *then*:

(a) $\text{man}_Q(D) \subseteq \text{man}_P(D)$;
(b) $\text{causes}_Q(M) \subseteq \text{causes}_P(M)$;
(c) *if D covers M in Q, then D covers M in P; and*
(d) *if D is an explanation for M in P and D covers M in Q, then D is an explanation for M in Q.*

COROLLARY 2.2. *Under the conditions of Lemma 2.1, if* $\mathbf{C}_P = \mathbf{C}_Q$, *then*:

(a) $\text{man}_p(D) = \text{man}_Q(D)$;
(b) $\text{causes}_p(M) = \text{causes}_Q(M)$;
(c) *D covers M in P if and only if D covers M in Q; and*
(d) *D is an explanation for M in P if and only if D is an explanation for M in Q.*

Using these results, we can now state fairly general conditions under which we can extract a portion of a diagnostic problem such that the extracted portion is itself a well-formed diagnostic problem.

LEMMA 2.3. *Let* $P = \langle \mathbf{D}_P, \mathbf{M}_P, \mathbf{C}_P, M_P^+ \rangle$ *be a diagnostic problem, and let* $\mathbf{D}_Q \subseteq \mathbf{D}_P$, $\mathbf{M}_Q = \mathrm{man}_P(\mathbf{D}_Q)$, $\mathbf{C}_Q = \mathbf{C}_P \cap (\mathbf{D}_Q \times \mathbf{M}_Q)$, *and* $M_Q^+ \subseteq \mathbf{M}_P$. *Then* $Q = \langle \mathbf{D}_Q, \mathbf{M}_Q, \mathbf{C}_Q, M_Q^+ \rangle$ *is a diagnostic problem if and only if* $\mathbf{D}_Q$ *is a cover for* M_Q^+ *in* P.

A general result like this is needed in analyzing the many calls to the recursive function Genset during concurrent problem solving (see line 7 in Solve, lines 15 and 16 in Genset, in Figure 1). If $P = \langle \mathbf{D}_P, \mathbf{M}_P, \mathbf{C}_P, M_P^+ \rangle$ is the initial diagnostic problem given as an argument to Solve, then each subsequent call to Genset involves $\mathbf{D}_Q \subseteq \mathbf{D}_P$ and $M_Q^+ \subseteq M_P^+ \subseteq \mathbf{M}_P$ as the first two arguments. The arguments $\mathbf{D}_Q$ and M_Q^+ in Genset[$\mathbf{D}_Q, M_Q^+, n$] thus represent a substructure of the original problem P, which we define formally as follows.

DEFINITION. Let $P = \langle \mathbf{D}_P, \mathbf{M}_P, \mathbf{C}_P, M_P^+ \rangle$ be a diagnostic problem, and let $\mathbf{D}_Q \subseteq \mathbf{D}_P$ and $M_Q^+ \subseteq M_P^+$. Then *the problem represented by* $\mathbf{D}_Q$ *and* M_Q^+, written prob($\mathbf{D}_Q, M_Q^+$), is

$$\mathrm{prob}\big(\mathbf{D}_Q, M_Q^+\big) = \langle \mathbf{D}_Q, \mathbf{M}_Q, \mathbf{C}_Q, M_Q^+ \rangle$$

where $\mathbf{M}_Q = \mathrm{man}_P(\mathbf{D}_Q)$ and $\mathbf{C}_Q = \mathbf{C}_P \cap (\mathbf{D}_Q \times \mathbf{M}_Q)$.

By Lemma 2.3 prob($\mathbf{D}_Q, M_Q^+$) is a well-defined diagnostic problem if and only if $\mathbf{D}_Q$ is a cover for M_Q^+ in P. Furthermore, the following holds.

LEMMA 2.4. *Let* $P = \langle \mathbf{D}_P, \mathbf{M}_P, \mathbf{C}_P, M^+ \rangle$ *be a diagnostic problem,* $D \subseteq \mathbf{D}_P$ *a cover for* M^+ *in* P, *and* $Q = \mathrm{prob}(D, M^+)$. *Then* $\forall D' \subseteq D$, $\mathrm{man}_Q(D') = \mathrm{man}_P(D')$.

In the following, we will therefore omit the subscript to "man" when Lemma 2.4 applies.

We now turn to establishing the correctness of function Genset by first showing that its two recursive calls (lines 15, 16) work as described earlier in this section. The first call to Genset[scope-I, manifs, n] in line 15 finds a generator set F for all explanations in Sol(prob(scope-I, manifs)) of size n (assuming the prob(scope-I, manifs) is a diagnostic problem of order n). From the previous lemmas it can be shown that this is the same as all explanations in Sol(prob (scope, manifs)) of size n which do not contain an element of I. If F is assigned $\varnothing$, it is either because prob(scope-I, manifs) is not a diagnostic problem, or because its order is greater than n; in either of these cases there are no explanations in Sol(prob(scope, manifs)) of size n that do not include an element of I. This result, formalized in the proof of correctness of Genset, is based on the following proposition.

LEMMA 2.5. *Let* $R = \langle \mathbf{D}_R, \mathbf{M}_R, \mathbf{C}_R, M^+ \rangle$ *be a diagnostic problem,* $D \subseteq \mathbf{D}_R$ *a cover for* M^+ *in* R, $P = \text{prob}(D, M^+)$, *and* $n = \text{order}(P)$. *Let* $d \in D$, $I = \{d' \in D | \text{man}^+(d') = \text{man}^+(d)\}$, *and* $S_I' = \{E \in \text{Sol}(P) | E \cap I = \varnothing\}$. *Let* $Q = \text{prob}(D - I, M^+)$.

(a) *If* $S_I' \neq \varnothing$, *then* Q *is a diagnostic problem,* $\text{order}(Q) = n$, *and* $\text{Sol}(Q) = S_I'$.
(b) *If* $S_I' = \varnothing$, *then either* Q *is not a diagnostic problem or* $\text{order}(Q) > n$.

The second recursive call to Genset[scope-I, manifs-man(d), $n-1$] in line 16 finds a generator set H for all explanations in Sol(prob(scope-I, manifs-man(d))) of size $n-1$. Generators in H are then composed with the generator (I) in line 17 to form G. It can be shown that G is the generator set for all explanations in Sol(prob(scope, manifs)) of size n which include an element of I. If prob(scope-I, manifs-man(d)) is of order n, then $H = \varnothing$, so G will also be empty, indicating that there are no explanations in Sol(prob(scope, manifs)) of size n which include an element of I. This argument, which forms part of the proof that Genset is correct, is based on the following proposition.

LEMMA 2.6. *Let* $R = \langle \mathbf{D}_R, \mathbf{M}_R, \mathbf{C}_R, M^+ \rangle$ *be a diagnostic problem,* $D \subseteq \mathbf{D}_R$ *a cover for* M^+ *in* R, $P = \text{prob}(D, M^+)$, *and* $n = \text{order}(P)$. *Let* $d \in D$, $I = \{d' \in D | \text{man}^+(d') = \text{man}^+(d)\}$, *and* $S_I = \{E \in \text{Sol}(P) | E \cap I \neq \varnothing\}$. *Let* $Q = \text{prob}(D - I, M^+\text{-man}(d))$. *Then:*

(a) Q *is a diagnostic problem.*

(b) *If* $S_I \neq \varnothing$, *then*

(1) $\text{order}(Q) = n - 1$, *and*
(2) $G = \{H_i \cdot (I) | H_i \in H\}$ *is a generator set for* S_I *whenever* $H = \{H_1, H_2, \ldots, H_k\}$ *is a generator set for* $\text{Sol}(Q)$.

(c) *If* $S_I = \varnothing$, *then* $\text{order}(Q) = n$.

From Lemmas 2.5 and 2.6, it can be shown that the set $G \cup F$ returned from Genset[scope, manifs, n] at line 18 is a generator set for Sol(prob(scope, manifs)) whenever prob(scope, manifs) is a diagnostic problem of order n. This result, formalized in Theorem 2.7 below, is then used to prove the correctness of function Solve in Theorem 2.8.

THEOREM 2.7 (Correctness of Genset). *Let a diagnostic problem* $\langle \mathbf{D}, \mathbf{M}, \mathbf{C}, M^+ \rangle$ *be given, and let* scope $\subseteq \mathbf{D}$, manifs $\subseteq M^+$ *and* N *be a nonnegative integer. Let* $R = \text{prob(scope, manifs)}$. *If* R *is a diagnostic problem of order* N, *then* Genset[scope, manifs, N] *terminates and returns a generator set for* $\text{Sol}(R)$. *If* R *is not a diagnostic problem, or if* $\text{order}(R) > N$, *then* Genset[scope, manifs, N] *terminates and returns* $\varnothing$.

THEOREM 2.8 (Correctness of Solve). *Let* P *be any diagnostic problem. Then* Solve[P] *will terminate and return a generator set for* $\text{Sol}(P)$.

```
 1.  Function  HT[D,M,C]
 2.       variables  n  integer, m  manifestation,
            manifs  manifestation-set, scope  disorder-set,
            hypothesis  generator-set;
 3.       begin
 4.         n := 0;                          (* Initially assume that order(P)=0,     *)
 5.         manifs := Ø;                          (* that no manifestations are known,  *)
 6.         scope := Ø;                            (* so no disorders have been evoked, *)
 7.         hypothesis := {Ø};  (* and hypothesize a generator set for Sol(P) = {Ø}.  *)
 8.         while  Moremanifs  do                 (* while another manifestation exists *)
 9.           m := Nextman;                             (* call the new manifestation m *)
10.           manifs := manifs ∪ {m};                   (* augment manifs known present *)
11.           scope := scope ∪ causes(m);                    (* augment evoked disorders *)
12.           hypothesis := an element from hypothesis/causes(m),
                                                                 (* update hypothesis *)
13.           if  hypothesis = Ø              (* no explanations for manifs of size n *)
14.             then  n=n+1;                            (* therefore increment order(P) *)
15.                   hypothesis:= Genset[scope,manifs,n]
                                                        (* and reconstruct hypothesis *)
16.           endif;
17.         endwhile
18.         return  hypothesis                               (* generator set for Sol(P) *)
19.       end
```

Fig. 2. Formal algorithm for solving sequential diagnostic problems. Reserved words are in bold type. Comments are enclosed in (*...*).

SEQUENTIAL PROBLEM SOLVING

In sequential problem solving one is given a diagnostic problem P where M^+ is *not* completely known at first. The goal is to discover M^+ and derive a generator set for Sol(P) simultaneously. This situation corresponds to real-world diagnostic problem solving where the diagnostician is given an initial set of manifestations that are present, and uncovers additional manifestations while accounting for the initial ones. As noted earlier, human diagnosticians use a sequential hypothesize-and-test approach during this process.

Figure 2 presents a formalized algorithm HT, for "hypothesize-and-test," which solves sequential diagnostic problems formulated in the GSC model. Given a partially specified diagnostic problem $P = \langle \mathbf{D}, \mathbf{M}, \mathbf{C}, ?\rangle$, function HT is called with three arguments: **D**, **M** and **C**. Although these arguments do not appear directly in the body of HT, they are used implicitly at a number of points [e.g., the reference to causes(m) in line 11]. As explained below, HT eventually discovers M^+ and returns a generator set for Sol(P).

The following data structures are used in HT:

n = the presumed value of order(P) at any point during problem solving;

m = a newly discovered manifestation;

manifs = the set of manifestations known to be present so far;

scope = causes(manifs), the set of all disorders d_i for which at least one manifestation is already known to be present; and

hypothesis = a generator set for the solution of prob(scope, manifs); i.e., the solution for those manifestations already known to be present, assuming (perhaps falsely) that no additional manifestations will be discovered subsequently. This represents the function HT's tentative or hypothesized solution at any point during problem solving, and in the end is returned as Sol(P) at line 18.

The initial assumption used in HT (lines 4–7) is that $n = \text{order}(P) = 0$, with manifs $= \varnothing$, scope $=$ causes(manifs) $= \varnothing$, and hypothesis $= \{\varnothing\}$, representing a generator set for the solution to prob($\varnothing, \varnothing$).

The rest of the body of HT is a single **while** loop on lines 8–17 whose execution is controlled by Moremanifs (line 8). Moremanifs is a predicate which, when called, returns "True" if additional manifestations exist in M^+ that are not already in manifs, and "False" if no additional manifestations remain undiscovered. The function Nextman then assigns one such manifestation as the value of the variable m (line 9). Moremanifs and Nextman roughly correspond to question generation in diagnostic expert systems. Our specification of Nextman is nondeterministic: we do not specify at present how or in what order the elements of M^+ are discovered (more on this later). Our only specific requirement is that when Moremanifs returns "False", *all* elements of M^+ have been found by Nextman.

As each manifestation m_j that is present is discovered by Nextman and assigned to m (line 9), manifs is updated simply by adding m_j to it (line 10). Scope is augmented to include any possible causes d_i of m_j which are not already contained in it (line 11). Finally, hypothesis is adjusted to accommodate m_j (lines 12–16).

The key step in this process is the adjustment to the tentative hypothesis in lines 12–16. The generator set representing the old hypothesis is divided by causes(m_j) at line 12, and an element of this division is taken to be the new hypothesis (division of a generator set by a set of disorders was defined after Lemma 1.14 in Part I). Recall that division of a generator set G by a nonempty set of disorders D produces a generator set H such that $[H] = \{E \in [G] | E \cap D \neq \varnothing\}$ (Part I, Lemmas 1.19 and 1.20). Thus, the new hypothesis created in line 12 represents exactly those explanations represented by the old hypothesis that can also cover the new m_j. If hypothesis $= \varnothing$ (line 13), then it must be because no previous explanations of size n can cover the new manifs (which includes m_j). We shall shortly prove that in this situation, explanations for the new manifs must be of cardinality $n+1$. Thus, n is incremented in line 15 and a generator set representing all explanations for the new manifs is derived in line 15.

A brief example should clarify the process used in function HT. Recall the example diagnostic problem P_1 given in Part I, where $M^+ = \{m_1, m_4, m_5\}$.

Events in order of their discovery by Nextman	n	manifs	scope	hypothesis
initially	0	$\varnothing$	$\varnothing$	$\{\varnothing\}$
m_1 present	1	$\{m_1\}$	$\{d_1,d_2,d_3,d_4\}$	$\{(\{d_1,d_2,d_3,d_4\})\}$
m_4 present	1	$\{m_1,m_4\}$	$\{d_1,d_2,d_3,d_4,d_5,d_8\}$	$\{(\{d_1,d_2\})\}$
m_5 present	2	$\{m_1,m_4,m_5\}$	$\{d_1,d_2,d_3,d_4,$ $d_5,d_7,d_8,d_9\}$	$\{(\{d_1,d_2\},\ \{d_7,d_8,d_9\}),$ $(\{d_8\},\ \{d_3,d_4\})\}$

Fig. 3. Sequential problem solving with the GSC model.

Suppose that the sequence of events occurring during problem solving is as illustrated in Figure 3. (The order in which manifestations are "discovered" here is arbitrary; any other order would have resulted in the same *final* hypothesis.)

Initially, manifs and scope are empty, $n = \text{order}(P)$ is taken to be zero, and the corresponding hypothesis $\{\varnothing\}$ contains a single generator $\varnothing$ which represents the solution to prob$(\varnothing, \varnothing)$ (first line in Figure 3). This corresponds to the state of function HT after execution of lines 4–7. When m_1 is discovered to be present by Nextman (line 9 of HT), m_1 is added to manifs, and the new scope is the union of the old scope with causes(m_1). Since hypothesis $= \{\varnothing\}$, division of hypothesis by causes(m_1) gives $\varnothing$. In lines 14–15 of the function HT, n is then incremented and Genset assigns to hypothesis a generator set representing Sol(prob$(\{d_1, d_2, d_3, d_4\}, (m_1\})$). Thus, at the end of one pass through the **while** loop of HT, the variables have the values shown on line 2 of Figure 3. The hypothesis shown here contains a single generator $(\{d_1, d_2, d_3, d_4\})$ and thus tentatively postulates that there are four possible explanations assuming $M^+ = \{m_1\}$, any one of which consists of a single disorder. The hypothesis asserts that "d_1 *or* d_2 *or* d_3 *or* d_4 is present."

On the next pass through the **while** loop m_4 is discovered to be present, so manifs and scope are augmented appropriately. Using generator division, a new hypothesis is developed, which represents the intersection of causes(m_4) with the single set in the only preexisting generator set in the hypothesis. Note that the resulting generator $(\{d_1, d_2\})$ represents precisely all explanations for the new manifs. This new hypothesis also illustrates another important point. As information about each possible manifestation becomes available, the hypothesis typically changes incrementally with a decrease in the number of explanations it represents (with the exception of situations where the hypothesis becomes empty). Line 3 in Figure 3 shows the variable values after the second pass through the **while** loop in HT.

When m_5 is discovered to be present on the next pass through the **while** loop, manifs and scope are again adjusted appropriately. However, in this case division of hypothesis by causes(m_5) makes hypothesis $= \varnothing$ because none of the previous explanations represented by the old hypothesis can now cover all known manifestations. Thus, n is incremented and Genset is called. At the end of the third pass through the **while** loop the variable values are as shown on lines 4 and 5 of Figure 3.

When Moremanifs is called again, it returns "False," as no further manifestations exist, so the **while** loop is not executed again. HT returns the final hypothesis (line 4 and 5 of Figure 3), which represents Sol(P_1), as was demonstrated in Part I.

Having presented the algorithm HT, we now direct our attention towards demonstrating its correctness. We begin by establishing that the hypothesized order of a diagnostic problem remains the same or increases by one as each new manifestation is incorporated into those known to be present. (The subsequent three lemmas could be stated more generally by permitting $m \in M^+$, but we restrict $m \notin M^+$ to emphasize the use of these results in sequential problem solving.)

LEMMA 2.9 (Incremental order increase). *Let $P = \langle \mathbf{D}, \mathbf{M}, \mathbf{C}, M^+ \rangle$ be a diagnostic problem with* order$(P) = n$, *and let $m \in \mathbf{M}$ such that $m \notin M^+$. Then $Q = \langle \mathbf{D}, \mathbf{M}, \mathbf{C}, M^+ \cup \{m\} \rangle$ is a diagnostic problem of order n or $n+1$.*

The following two lemmas show how generator division can be used to update a hypothesis whenever a new manifestation is discovered during sequential problem solving.

LEMMA 2.10. *Let $P = \langle \mathbf{D}, \mathbf{M}, \mathbf{C}, M^+ \rangle$ be a diagnostic problem of order n, and let $G = \{G_1, G_2, \ldots, G_N\}$ be a generator set such that $[G] = \mathrm{Sol}(P)$. Let $m \in \mathbf{M}$-M^+, and let $H \in G/$causes(m). Then H is a generator set with $[H] = \{E \in [G] \mid E$ covers $m\}$.*

LEMMA 2.11. *Let P, G, m, and H be as in Lemma 2.10, and let $Q = \langle \mathbf{D}, \mathbf{M}, \mathbf{C}, M^+ \cup \{m\} \rangle$.*

(a) *If $H = \varnothing$, then Q is a diagnostic problem of order $n+1$.*
(b) *If $H \neq \varnothing$, then Q is a diagnostic problem of order n, and $[H] = \mathrm{Sol}(Q)$.*

We can now proceed to the analysis of algorithm HT. We adopt the following notational convenience.

DEFINITION. If V is a variable used in algorithm HT, then if $i = 1, \ldots, |M^+|$, V_i is the value of V after the ith execution of the **while** loop; and V_0 is the value of V just before the first execution of the **while** loop. Similarly, $P_i =$ prob(scope$_i$, manifs$_i$) is the problem represented by scope and manifs after the ith execution of the **while** loop, and $P_0 =$ prob(scope$_0$, manifs$_0$) $=$ prob($\varnothing$, $\varnothing$)

is the problem represented by scope and manifs just before the first execution of the **while** loop.

LEMMA 2.12. *For* $i = 0, \ldots, |M^+|$:

(a) $\text{manifs}_i = \{m_1, m_2, \ldots, m_i\}$, *where* m_i *is the ith manifestation returned by* Nextman;

(b) $\text{scope}_i = \text{causes}(\text{manifs}_i)$; *and*

(c) P_i *is a diagnostic problem.*

The major results of this section can now be stated and proved, verifying the correctness of algorithm HT.

THEOREM 2.13. *For* $i = 0, \ldots, |M^+|$, $n_i = \text{order}(P_i)$ *and* $[\text{hypothesis}_i] = \text{Sol}(P_i)$.

COROLLARY 2.14. *Let* $P = \langle \mathbf{D}, \mathbf{M}, \mathbf{C}, M^+ \rangle$ *be a diagnostic problem. Then* HT$[\mathbf{D}, \mathbf{M}, \mathbf{C}]$ *terminates and returns a generator set for* $\text{Sol}(P)$.

APPLICATIONS TO DIAGNOSTIC EXPERT SYSTEMS

One of the purposes of the GSC formalization is to provide a framework that captures the basic idea of abduction as used in recent diagnostic expert systems which often claim to model human abductive reasoning. This section therefore briefly illustrates the relationship between some of these computational models and the formal GSC model presented here.

In contrast to the GSC model, most diagnostic expert systems that use a hypothesize-and-test inference mechanism or which might reasonably be considered as models of diagnostic reasoning depend heavily upon the use of production rules (e.g., [17, 32]). These systems use a hypothesis-driven approach to guide the invocation of rules, which in turn modify the hypothesis. While a rule-based hypothesize-and-test approach may produce good performance, such a process does not provide a convincing model of what has been learned about human diagnostic reasoning in the empirical studies cited in Part I. Furthermore, invocation of rules to make deductions or perform actions does not capture in a general sense such "abductive" concepts as coverage, parsimony, or explanation as defined earlier.

In contrast to these rule-based systems, several recent computational models of diagnostic reasoning are more "purely abductive" in nature. Three of these systems, namely INTERNIST, KMS.HT, and IDT, will be considered further here. The goal is to illustrate how the GSC model formalizes the basic abductive aspects of inference used in these systems, and to show in what ways they differ from the GSC model. We divide this comparative analysis into two parts based on whether multiple or single hypotheses are assumed.

MULTIPLE-HYPOTHESES ASSUMPTION

A number of abductive diagnostic expert systems assume that multiple disorders or hypotheses may occur simultaneously, and we consider two of these here: INTERNIST and KMS.HT. INTERNIST is a large medical expert system developed and evaluated earlier by others on an intuitive basis [12, 36]. INTERNIST uses diagnostic knowledge organized in a descriptive fashion and does not rely on production rules to guide its hypothesize-and-test process. It is based on roughly the same principles as the GSC model: it attempts to account for M^+ with the minimum number of disorders. In contrast to the GSC model, however, INTERNIST makes the assumption that the equivalent of a single generator is sufficient to represent the solution to a diagnostic problem. As we have seen (e.g., Figure 3), this is not always the case, raising the possibility that INTERNIST might at times omit some relevant explanations. Furthermore, the INTERNIST inference mechanism uses a heuristic scoring procedure to guide the construction and modification of the single generator it constructs. This process is essentially serial or depth-first, unlike the more parallel or breadth-first approach in the GSC model. INTERNIST first attempts to establish the presence of one disorder and then proceeds to establish others. This roughly corresponds to constructing and completing a single set of disorders in a generator in the GSC model, and then later returning to construct the additional sets for the generator. The serial approach used by INTERNIST proved to be a significant limitation when dealing with real-world diagnostic problems [12], reflecting its "inability to perceive the multiplicity of problems in a case all at once" [37].

In addition, using the GSC model to analyze the INTERNIST approach to grouping together competing disorders (those disorders forming a set in the generator it constructs) shows that even in some situations where a single generator *could* generate the entire solution, INTERNIST will apparently fail to find this generator. This is explained below.

INTERNIST uses the following simple but clever heuristic to group competing disorders together: "Two diseases are competitors if the items not explained by one disease are a subset of the items not explained by the other; otherwise, they are alternatives (and may possibly coexist in the patient)" [12]. In terms of the GSC model, this corresponds to stating that d_1 and d_2 are competitors if M^+-man(d_1) contains or is contained in M^+-man(d_2). Suppose that M^+-man$(d_1) \subseteq M^+$-man(d_2). Then it follows that $\text{man}^+(d_2) \subseteq \text{man}^+(d_1)$, so the competing-disorders theorem (Part I, Theorem 1.8) applies to d_1 and d_2. This makes it quite reasonable to intuitively view d_1 and d_2 as "competitors" in the GSC model, and to group them together as is done in INTERNIST. However, the GSC model also shows that there are situations where the INTERNIST heuristic would fail to group together disorders as competitors which should clearly be considered as such if one is using a generator to represent the solution to a diagnostic problem.

For example, suppose $M^+ = \{m_1, m_2, \ldots, m_8\}$ and only d_1, d_2, and d_3 are in causes(M^+), so scope $= \{d_1, d_2, d_3\}$. Suppose $\text{man}^+(d_1) = \{m_2, m_4, m_5, m_6, m_7, m_8\}$, $\text{man}^+(d_2) = \{m_3, m_4, m_5, m_6, m_7, m_8\}$, and $\text{man}^+(d_3) = \{m_1, m_2, m_3\}$. In the GSC model, Sol($P$) $= \{\{d_1, d_3\}, \{d_2, d_3\}\}$, which can be represented by the single generator $(\{d_1, d_2\}, \{d_3\})$, where d_1 and d_2 are grouped together as competitors. Suppose that d_1 was ranked highest by the INTERNIST heuristic scoring procedure. Then M^+-man$(d_1) = \{m_1, m_3\}$, and M^+-man$(d_2) = \{m_1, m_2\}$, neither of which is contained in the other. Thus, in this situation where a single generator *could* correctly represent the solution, INTERNIST would apparently fail to group d_1 and d_2 together as competitors (exactly what INTERNIST would do in this situation is unclear).

INTERNIST has introduced many innovative and influential concepts for building diagnostic expert systems, and our comments above should not be viewed as a criticism of the pioneering effort which it represents. Rather, our comments are intended to reinforce the validity of the formal GSC model by showing that its results correspond to those arrived at independently by others on an intuitive basis. In addition, the comments above illustrate the utility of a theoretical foundation for analyzing current computational models, so that a careful assessment of their limitations can be made.

The second computational model of diagnostic reasoning assuming the possibility of multiple simultaneous disorders is a more recent domain-independent system called KMS.HT [21]. The inference mechanism in KMS.HT essentially *is* algorithm HT (Figure 2) with a variety of enhancements designed to make this approach more robust in the real world (see below). KMS.HT antedated by years and in part motivated the development of the formalization in the GSC model.

The following paragraphs provide some examples of the kinds of enhancements to the GSC model that exists in KMS.HT. The reader should keep in mind, however, that in spite of these embellishments the fundamental driving force in KMS.HT is a sequential GSC-like approach to diagnostic problem solving. It is in this sense that the GSC model is an abstraction of an implemented expert system such as has been advocated by Nilsson [16]. Example real-world expert systems developed with KMS.HT are described elsewhere (see [21]).

One type of enhancement to the function HT as it is described in Figure 2 (HT also calls Genset, Figure 1) is to make the nondeterministic steps deterministic in a reasonably efficient manner. For example, in Genset the nondeterministic selection of $d \in$ scope (line 13, Figure 1) is based on the following criterion (simplified):

> If there exists $m \in$ manifs that is pathognomonic for some $d' \in$ scope, let $d = d'$. Otherwise, use $d \in$ scope such that $|\text{man}(d) \cap \text{manifs}|$ is maximized.

The reason for selecting a d' which causes a pathognomonic manifestation is that by Part I, Lemma 1.23, such a d' must be in any explanation for manifs. Furthermore, by Part I, Theorem 1.24, the recursive call to Genset in line 15 of

Genset can therefore be omitted and F assigned $\varnothing$, potentially a significant computational saving. The alternative, selecting $d \in$ scope that maximizes $|\text{man}(d) \cap \text{manifs}|$, is a useful heuristic because during real-world problem solving this alternative usually (but not always) selects a disorder that will be in an explanation for manifs, thus making the call to Genset in line 16 likely to be successful.

Another enhancement made to the algorithm HT as it is used in KMS.HT is the partitioning of diagnostic problems into independent subproblems using the concept of connected manifestations introduced in Part I. Each subproblem has its own manifs, scope, and hypothesis. Perhaps surprisingly, constructing and maintaining independent subproblems in this fashion turn out to be relatively easy. Whenever a new manifestation m_i is found to be present, the set causes(m_i) is intersected with the scope of each preexisting subproblem. When this intersection is nonempty, we will say that m_i is *related* to the corresponding subproblem in that m_i is connected to at least one other manifestation in that subproblem. There are three possible results of identifying the subproblems to which m_i is related. First, m_i may not be related to any preexisting subproblem. In this case, a new subproblem is created with manifs $= \{m_i\}$, scope = causes(m_i), and hypothesis $= \{(\text{causes}(M_i))\}$, e.g., the second row in Figure 3. Second, m_i may be related to exactly one subproblem, in which case m_i is assimilated into that subproblem as dictated by algorithm HT (e.g., the last two rows in Figure 3). Finally, m_i may be related to more than one existing subproblem. In this situation, these previously independent subproblems are "joined" together. The manifs (scope) for the new subproblem is constructed by appending together the manifs (scope) sets of the related subproblems. The hypothesis for the new "joined" subproblem is constructed by first composing the generators from the hypotheses of the related subproblems and then dividing the resultant generator set by causes(m_i). At the end of the problem-solving process, the generator set for the solution to the entire original diagnostic problem is constructed by appropriately composing the generators from the subproblems. That this approach is correct follows directly from Lemma 1.25(b) and Theorem 1.26 of Part I; that it is reasonably efficient follows from the reduced order of each diagnostic subproblem and from the fact that generator composition in this context is relatively fast.

Several other issues are addressed and resolved in adopting the algorithm HT for use in KMS.HT, including question generation methods, termination criteria, ranking of competing disorders, and adjustment of the hypothesis in the context of information about nonmanifestations (such as a patient's age in medical problem solving). Generally, a heuristic approach is taken to resolving each of these issues. For example, the GSC model involves a nondeterministic approach to question generation and termination of the problem-solving process (line 8 and 9 in function HT, Figure 2). The only requirement is that when Moremanifs returns "False" in line 8, Nextman has already previously found all of M^+. In the KMS.HT implementation, Nextman is a heuristically guided function of the current hypothesis, resulting in focused and directed questioning of the user. The current implementation of Nextman in KMS.HT is such that it might not discover all of M^+ by the time the termination criteria are reached (a knowledge-base author can alter this), although this would be very easy to change. For further details on these issues and others, as well as example expert systems built with KMS.HT, the interested reader is referred to [21].

SINGLE-HYPOTHESIS ASSUMPTION

In some diagnostic problems of more limited scope it proves convenient and appropriate to make the single-hypothesis assumption: that only one disorder can occur at a time. IDT, for intelligent diagnostic tool, is an expert system for diagnosing faults in PDP 11/03 computers that makes this assumption [27]. In IDT a computer is viewed as composed of a number of "atomic units" (controllers, interfaces, disk drives, etc.). A disorder d_i represents the hypothesis that the ith atomic unit is broken. Diagnostic knowledge is represented as deductive formulas associated with test results, e.g., "$\sim d_1 \to d_2 \vee d_3$" or "$d_4 \to \sim d_6$." These formulas are not restricted to be Horn clauses. The goal of the diagnostic process is to prove a formula of the form "$d_i \vee d_j \vee \cdots \vee d_k$," where the members of the set $\{d_i, d_j, \dots, d_k\}$ are all inside a single replaceable computer module. The developers of IDT prove that if one makes the "extralogical single-fault assumption," then *any* formula E associated with a test result can be transformed into a disjunctive formula (a clause) containing only positive literals (nonnegated d_i) that is equivalent to E [27]. Furthermore, if one represents each such clause "$d_1 \vee d_2 \vee \cdots \vee d_n$" by its corresponding set of literals $\{d_1, d_2, \dots, d_n\}$, then it can be shown that set intersection is the only operation necessary for combining test results [27].

From the perspective of the GSC model, each IDT test result corresponds to a manifestation m, and the set of positive literals associated with that test result corresponds to causes(m). Furthermore, the inference mechanism used in IDT, namely set intersection, can be shown to be a special case of generator division that occurs when the single-hypothesis assumption is made. This can be stated as the following

PROPOSITION 2.15. *Set intersection is a special case of generator set division. More precisely, let* $P = \langle \mathbf{D}, \mathbf{M}, \mathbf{C}, M^+ \rangle$ *be a diagnostic problem, where* order(P) $= 1$. *Then the sequence of steps*

1. $G := \{(\mathbf{D})\}$;
2. **for** $m \in M^+$ **do** $G :=$ *a division of G by* causes(m);

produces a generator set $G = \{(D')\}$, *where* $D' = \bigcap_{m \in M^+}$ causes(m).

Thus, the inference mechanism in IDT, roughly captured in the above steps, can be viewed as a special case of algorithm HT where order(P) = 1. Note that in line 1 of the above sequence, **D** could be replaced by causes(m) for any $m \in M^+$ and the proposition would still hold.

KMS.HT, described in the preceding section, can also operate in a special mode where the single hypothesis assumption holds. In this mode two trivial changes exist in algorithm HT (Figure 2):

(i) Line 7 is replaced with

$$\text{hypothesis} := \{(\mathbf{D})\}.$$

(ii) Lines 13–16 are replaced with

$$\textbf{if } \text{hypothesis} = \varnothing \textbf{ then error endif};.$$

Change (i) represents the initial hypothesis that exactly one disorder is present, i.e., that order(P) = 1. Change (ii) calls function **error** to generate an error message and return, reflecting that a violation of the single disorder assumption has occurred. In real-world expert systems built with KMS.HT, such an error might be caused by "noisy data," incorrectness of the single disorder assumption, or an errorful knowledge base (see [21] for further discussion). It should be evident to the reader that the modified algorithm HT is correct within the GSC model if the single-hypothesis assumption holds.

TOWARDS A GENERAL THEORY OF ABDUCTION

This and the previous paper have presented a formalization of the basic abductive reasoning processes used in a number of contemporary diagnostic expert systems. The definitions in Part I of a diagnostic problem, an explanation, and the solution to a diagnostic problem effectively represent the basic postulates of the GSC model. The validity of all subsequent theorems and the correctness of the algorithm HT are consequences of these fundamental definitions. As discussed above, the "results" of the GSC formalization capture in a domain-independent fashion many features of expert systems like INTERNIST, KMS.HT, and IDT. Further, the GSC model has provided insights into the methods used in such application-oriented programs that were not obvious in advance. For example, the recognition that there are situations where INTERNIST might not behave as expected is an important insight derived from the GSC model. The demonstration that the deductive formulas and set-intersection method used in IDT can be viewed as a special case of the GSC model (under the "single hypothesis" assumption) is also significant. To the extent that one accepts the position, suggested by others [33], that intersection search is of fundamental importance in knowledge-related areas of intelligence, it seems useful to explore any generalization of this concept.

Still, a great deal remains to be done to achieve a truly general theory of abductive diagnostic problem solving. We therefore outline below a number of issues awaiting exploration and resolution in this area, in the hope that such work will lead to more robust diagnostic expert systems in the future. Our intent is to clarify the limitations of the existing GSC model, and perhaps entice others to join in solving some of the difficult problems involved. Additional issues related to the basic GSC model are discussed first, and this is followed by an outline of an extended notion of "parsimonious covering" as a generalization of the GSC model.

FURTHER DEVELOPMENT OF THE EXISTING GSC *MODEL*

At present, there is clearly a need for further development and investigation of the existing GSC model. The issues of concern here do not require fundamental changes to the GSC model, but represent elaborations of the existing theory. For example, it would be valuable to determine optimal criteria for generating questions (i.e., criteria for selecting the next manifestation whose presence or absence is to be determined) and for deciding when to terminate sequential problem solving. The integration of probability theory within the GSC model to permit ranking of competing explanations would strengthen the model. Such an integration would permit more detailed analysis of scoring mechanisms in existing abductive expert systems and the possibility of identifying and handling noisy data in real-world systems. Also, it would be an important advance if criteria could be specified whereby the number of generators in a generator set could be minimized. At present, this is done heuristically in the KMS.HT implementation, and the authors know of some criteria which, when satisfied, permit a generator set to be collapsed into another generator set having fewer generators. Unfortunately, finding necessary and sufficient conditions to guarantee that there is no generator set of smaller cardinality for the solution has proven to be an elusive goal.

Another issue is the relationship between abductive inference as formalized in the GSC model and deductive inference as used in many rule-based expert systems. Some initial work has been done in this area [27, 34], but we suspect that the issues involved are just beginning to be understood. Particularly of interest is the relationship between the GSC model and recent theoretical developments in default reasoning and nonmonotonic logics. The GSC formalization clearly represents a nonmonotonic logic [38]. Each cycle through function HT involves making inferences in the context of incomplete information and under the default assumption that no additional manifestations exist. Whenever this assumption is found not to hold (a new manifestation is discovered), hypothesis revision occurs. Unlike deductive logics, inferences cannot be made "locally" (e.g., via modus ponens) but are inherently a global property of the causal relation **C**. Furthermore, unlike deductive models, there is no absolute "truth" in the GSC model, only a notion of plausibility [38]. A better understanding of these and related properties should lead to expert systems that effectively and correctly integrate deductive and abductive inference methods during problem solving.

Finally, we have recently investigated experimentally the support of answer justification methods in abductive expert systems [39]. A corresponding theoretical extension to the GSC model would enhance its generality.

PARSIMONIOUS COVERING: GENERALIZING THE GENERALIZATION

As noted in Part I, the GSC formulation makes certain simlifying assumptions. These include the assumptions that disorders are independent of one another, and that disorders and their associations with manifestations are of roughly equal likelihood. In the real world, such assumptions do not always hold, especially as one moves from restricted domains to more general diagnostic problems. Thus, for example, one disorder might directly or indirectly cause another, and therefore disorders would not be independent of one another. Or, as pointed out in Part I, there are situations where a nonminimal cover (e.g., consisting of two very common disorders) might be a more plausible explanation for a set of manifestations than a minimal cover (e.g., consisting of one very rare disorder).

To study these and related questions, we are currently investigating a significantly generalized formalization of the GSC model that we will refer to simply as "parsimonious covering" [35]. Within this framework, a diagnostic problem is formalized as a 4-tuple $\langle \mathbf{H}, \mathbf{C}, \mathbf{D}, H' \rangle$, where **H** is a finite nonempty set of entities (hypotheses), $\mathbf{C} \subseteq \mathbf{H} \times \mathbf{H}$ is a causality relation, $\mathbf{D} \subseteq \mathbf{H}$ is a distinguished set of "ultimate disorders," and H' is a distinguished set of entities said to be present. The set **H** includes both **D** and **M** of the GSC model as well as any intermediate abnormal states. H' corresponds to M^+ in the GSC model, but may also include volunteered information about preexisting abnormal states or disorders. A generalized notion of "covering" has been formalized that involves "causal chaining": situations where h_i causes h_j, and h_j causes h_k, so h_i causes h_k indirectly. Representations of parsimony other than minimality are under study, such as the notion of "irredundancy," which handles some situations involving nonminimal but highly plausible covers (a cover of H' is

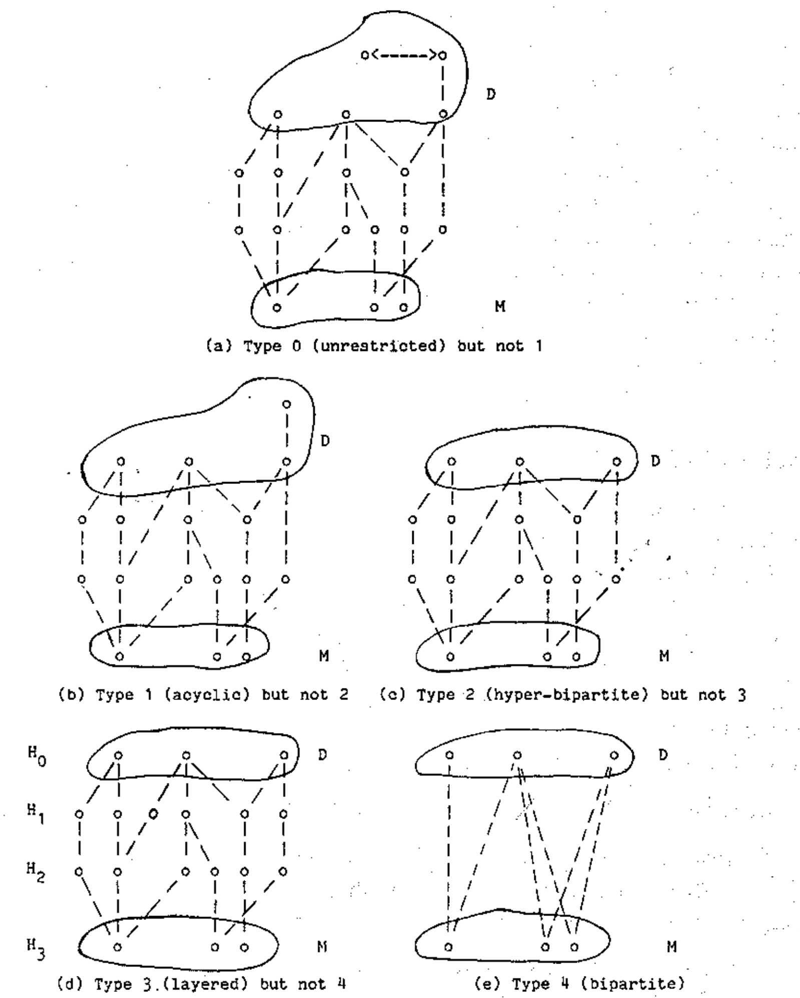

Fig. 4. A taxonomy of parsimonious covering models. (a) Type 0 (unrestricted) problems include all parsimonious models. (b) Type 1 (acyclic) problems are those where **C** imposes an acyclic digraph on **H**. (c) Type 2 (hyperbipartite) impose $\mathbf{D} \cap \mathbf{M} = \varnothing$ with no causal associations between ultimate disorders in **D**. (d) Type 3 (layered) problems partition **H** into disjoint layers with causal associations only between elements in adjacent layers. (e) Type 4 (bipartite) problems consist only of two layers.

irredundant if none of its proper subsets is also a cover of H'). Peng has proposed a taxonomy of diagnostic problems within the parsimonious covering framework [35], and this is illustrated in Figure 4. Note that the GSC model is a special case of Type IV problems, which are themselves the simplest class of parsimonious covering models.

Finally, we observe that the GSC model as an abstraction of abduction occupies the same level as propositional logic does with respect to deduction. The GSC model does not incorporate variables or quantification. We have done some preliminary experimental work with the use of variables in the GSC model [40], but the formal introduction of such concepts into an abductive logic remains an important area for future research.

APPENDIX. PROOFS

Lemma 2.1. (a): If $m \in \text{man}_Q(D)$, then there exists $d \in D$ such that $\langle d, m \rangle \in \mathbf{C}_Q$, so $\langle d, m \rangle \in \mathbf{C}_p$, and therefore $m \in \text{man}_p(D)$.

(b): Analogous to (a).

(c): Since D covers M in Q, we have $m \in \text{man}_Q(D)\ \forall m \in M$. Then by (a), $m \in \text{man}_P(D)\ \forall m \in M$, so D covers M in P.

(d): Suppose there is a cover D' for M in Q such that $|D'| < |D|$. Then by (c), D' covers M in P, contradicting that D is an explanation for M in P. Thus, there is no cover D' for M in Q such that $|D'| < |D|$, so since D covers M in Q, it follows that D is an explanation for M in Q.

Corollary 2.2. Immediate from Lemma 2.1.

Lemma 2.3. "If": Suppose D_Q is a cover for M_Q^+ in P. To show that Q is a diagnostic problem we show that (i) $D_Q = \text{domain}(\mathbf{C}_Q)$, (ii) $\mathbf{M}_Q = \text{range}(\mathbf{C}_Q)$, and (iii) $M_Q^+ \subseteq \mathbf{M}_Q$.

(i) Since $\mathbf{C}_Q = \mathbf{C}_P \cap (\mathbf{D}_Q \times \mathbf{M}_Q)$, it follows that $\text{domain}(\mathbf{C}_Q) \subseteq \mathbf{D}_Q$. Conversely, if $d \in \mathbf{D}_Q$, then $d \in \mathbf{D}_P$, so by the definition of a diagnostic problem there exists an $m \in \mathbf{M}_p$ such that $\langle d, m \rangle \in \mathbf{C}_p$. Thus $m \in \text{man}_p(d)$, so $m \in \text{man}_p(\mathbf{D}_Q) = \mathbf{M}_Q$. Hence, $\langle d, m \rangle \in \mathbf{C}_P \cap (\mathbf{D}_Q \times \mathbf{M}_Q) = \mathbf{C}_Q$, so $d \in \text{domain}(\mathbf{C}_Q)$, whereby $\mathbf{D}_Q \subseteq \text{domain}(\mathbf{C}_Q)$. Thus, $\mathbf{D}_Q = \text{domain}(\mathbf{C}_Q)$.

(ii) Since $\mathbf{C}_Q = \mathbf{C}_P \cap (\mathbf{D}_Q \times \mathbf{M}_Q)$, it follows that $\text{range}(\mathbf{C}_Q) \subseteq \mathbf{M}_Q$. Conversely, if $m \in \mathbf{M}_Q = \text{man}_P(\mathbf{D}_Q)$, then there exists $d \in \mathbf{D}_Q$ such that $\langle d, m \rangle \in \mathbf{C}_p$. But $\langle d, m \rangle \in \mathbf{D}_Q \times \mathbf{M}_Q$, so $\langle d, m \rangle \in \mathbf{C}_p \cap (\mathbf{D}_Q \times \mathbf{M}_Q) = \mathbf{C}_Q$. Hence, $m \in \text{range}(\mathbf{C}_Q)$ and therefore $\mathbf{M}_Q = \text{range}(\mathbf{C}_Q)$.

(iii) $M_Q^+ \subseteq \text{man}_p(\mathbf{D}_Q)$ because $\mathbf{D}_Q$ is a cover for M_Q^+ in P. Hence, $M_Q^+ \subseteq \mathbf{M}_Q$ by the conditions of this lemma.

"Only if": Suppose Q is a diagnostic problem. Then

$$\begin{aligned} M_Q^+ &\subseteq \mathbf{M}_Q && \text{by the definition of a diagnostic problem} \\ &= \text{man}_Q(\mathbf{D}_Q) && \text{by Part I, Lemma 1.1(d')} \\ &\subseteq \text{man}_p(\mathbf{D}_Q) && \text{by Lemma 2.1a.} \end{aligned}$$

Thus, $\mathbf{D}_Q$ is a cover for M_Q^+ in P.

Lemma 2.4. For any $d \in D'$,

$$\begin{aligned} \text{man}_Q(d) &= \{m \mid \langle d, m\rangle \in \mathbf{C}_Q\} && \text{by definition of man} \\ &= \{m \mid \langle d, m\rangle \in \mathbf{C}_p \cap (D \times \text{man}_p(D))\} && \text{by definition of prob} \\ &= \{m \mid \langle d, m\rangle \in \mathbf{C}_p\} \\ &\quad \cap \{m \mid \langle d, m\rangle \in D \times \text{man}_p(D)\} \\ &= \text{man}_p(d) \cap \text{man}_p(D) && \text{since } d \in D' \subseteq D \\ &= \text{man}_p(d) && \text{since } d \in D. \end{aligned}$$

Lemma 2.5. Note that by Lemma 2.3, P is a diagnostic problem.

(a): Suppose $S_I' \neq \varnothing$, and let $E \in S_I'$. Then by the definition of S_I', E is an explanation for M^+ in P and $E \in D - I$. Thus, $M^+ \subseteq \text{man}(E) \subseteq \text{man}(D - I) = \text{man}(\mathbf{D}_Q)$, so Q is a diagnostic problem by Lemma 2.3. But by Lemma 2.4, $\text{man}_Q(E) = \text{man}_p(E)$, so E covers M^+ in Q. Hence, by Lemma 2.1(d), E is an explanation for M^+ in Q, so $\text{order}(Q) = |E| = \text{order}(P) = n$. Also, since the above reasoning holds $\forall E \in S_I'$, it follows that $S_I' \subseteq \text{Sol}(Q)$. It remains to show that $\text{Sol}(Q) \subseteq S_I'$.

Let $E' \in \text{Sol}(Q)$. Then E' is an explanation for M^+ in Q, so by Lemma 2.1(c), E' covers M^+ in P. Then, since $\text{order}(P) = \text{order}(Q)$, E' is also an explanation for M^+ in P. But since $\text{domain}(\mathbf{C}_Q) = D - I$, we have $E' \subseteq D - I$, so $E' \cap I = \varnothing$ and thus $E' \in S_I'$. Hence, $\text{Sol}(Q) \subseteq S_I'$, so by the last line of the preceding paragraph, $\text{Sol}(Q) = S_I'$.

(b): Suppose $S_I' = \varnothing$. By Lemma 2.3, Q is a diagnostic problem if and only if $M^+ \subseteq \text{man}_p(D - I)$.

Case 1: $M^+ \not\subseteq \text{man}_p(D - I)$. Then Q is not a diagnostic problem.

Case 2: $M^+ \subseteq \text{man}_p(D - I)$. Then Q is a diagnostic problem. Let $E \in \text{Sol}(Q)$. Since $\text{domain}(\mathbf{C}_Q) = D - I$, we have $E \cap I = \varnothing$, so $E \notin \text{Sol}(P)$ because

$S_I' = \varnothing$. But by Lemma 2.1(c), E covers M^+ in P because $\mathbf{C}_Q \subseteq \mathbf{C}_p$. Thus it must be that $|E| > \text{order}(P) = n$. Hence, $\text{order}(Q) = |E| > n$.

Lemma 2.6. (a): By Lemma 2.3, P is a diagnostic problem, since $D \subseteq \mathbf{D}_R$ is a cover for M^+. We can also use Lemma 2.3 to show that Q is always a diagnostic problem by proving that $D - I$ covers M^+-man(d). First note that since P is a diagnostic problem, D covers M^+ by Lemma 1.1 and the definition of a diagnostic problem. Thus, $M^+ \subseteq \text{man}(D)$. Also, $\text{man}^+(d') = \text{man}^+(d)$ $\forall d' \in I$, so $\text{man}^+(I) = \text{man}^+(d)$. Thus

$$
\begin{aligned}
M^+\text{-man}(d) &= M^+\text{-man}^+(d) && \text{by definition of man}^+\\
&= M^+\text{-man}^+(I) && \text{by the above}\\
&= M^+\text{-man}(I) && \text{by definition of man}^+\\
&\subseteq \text{man}(D)\text{-man}(I) && \text{by the above}\\
&\subseteq \text{man}(D - I) && \text{by Part I, Lemma 1.1(f).}
\end{aligned}
$$

Since $D - I$ covers M^+-man(d), Q is a diagnostic problem by Lemma 2.3.

(b): Suppose $S_I \neq \varnothing$. Then there is a explanation E for M^+ in P and a $d' \in I$ such that $d' \in E$. Note that $|E| = n$, and E contains no other member of I other than d' by Part I, Theorem 1.8(a). Thus $E - \{d'\} \subseteq D - I = \mathbf{D}_Q$.

(1) It follows that

$$
\begin{aligned}
M_Q^+ &= M^+\text{-man}(d)\\
&\subseteq \text{man}^+(E)\text{-man}(d) && \text{since } E \text{ covers } M^+ \text{ in } P\\
&= \text{man}^+(E)\text{-man}^+(d) && \text{by definition of man}^+\\
&= \text{man}^+(E)\text{-man}^+(d') && \text{by definition of } I\\
&\subseteq \text{man}^+(E - \{d'\}) && \text{by Part I, Lemma 1.1(f).}
\end{aligned}
$$

Thus, $E - \{d'\}$ covers M_Q^+. Furthermore, $E - \{d'\}$ is a minimal cover, for if there were a smaller cover D', $D' \cup \{d'\}$ would cover M^+ [by Lemma 2.1(c)], contradicting the minimality of E in P. Thus, $E - \{d'\}$ is an explanation for $M^+\text{-man}_p(d) = M_Q^+$ in Q, so $\text{order}(Q) = |E - \{d'\}| = n - 1$.

(2) Let G and H be as defined in the statement of the lemma. To show that G is a generator set, we must show that each $H_i \cdot (I)$ is a generator, and that $[H_i \cdot (I)] \cap [H_j \cdot (I)] = \varnothing \ \forall i \leq j$.

Note that (I) and each H_i are generators, and by the definition of Q, none of the H_i can contain any member of I. Thus, by the definition of composition, $H_i \cdot (I)$ is a generator for each i. Furthermore, if $i \leq j$ then

$$[H_i \cdot (I)] \cap [H_j \cdot (I)] = \{E \cup \{d'\} | E \in [H_i] \cap [H_j] \text{ and } d' \in I\} \qquad \text{by Part I, Lemma 1.14}$$

$$= \{E \cup \{d'\} | E \in \varnothing \text{ and } d' \in I\} \quad \text{because } H \text{ is a generator set}$$

$$= \varnothing.$$

It remains to show that $[G] = S_I$. To show that $[G] \subseteq S_I$, let $E \in [G]$. Then $E \in [H_i \cdot (I)]$ for some i, so $E = D' \cup \{d'\}$, where $D' \in [H_i] \subseteq \text{Sol}(Q)$ and $d' \in I$. Thus $M^+\text{-man}(d) = M_Q^+ \subseteq \text{man}(D')$, so $M^+ \subseteq \text{man}(D') \cup \text{man}(d) = \text{man}(D') \cup \text{man}(d') = \text{man}(D' \cup \{d'\}) = \text{man}(E)$, so E covers M^+. But $|E| = |D'| + 1 = \text{order}(Q) + 1 = n$. Thus E is an explanation for M^+ in P, so $E \in S_I$. Therefore, $[G] \subseteq S_I$.

To show that $S_I \subseteq [G]$, let $E \in S_I$, and let $d' \in E \cap I$. From the proof of (1) above we see that $E - \{d'\}$ is an explanation for M_Q^+, so $E - \{d'\} \in [H_i]$ for some i. Thus $E \in [H_i \cdot (I)]$, so $E \in [G]$ and $S_I \subseteq [G]$. Combined with the result from the previous paragraph, this implies that $[G] = S_I$.

(c): Suppose $S_I = \varnothing$, and let $E \in \text{Sol}(P)$. Then $E \cap I = \varnothing$, so $E \subseteq D - I = \mathbf{D}_Q$. Thus, $M^+\text{-man}(d) \subseteq \text{man}(E)$, so E covers $M^+\text{-man}(d)$. Hence, if D' is an explanation for $M^+\text{-man}(d)$, then $|D'| \leq |E| = n$. Furthermore, if D' is an explanation for $M^+\text{-man}(d)$, then

$$M^+ \subseteq [M^+\text{-man}(d)] \cup \text{man}(d)$$

$$\subseteq \text{man}(D') \cup \text{man}(d) \qquad \text{since } D' \text{ covers } M_Q^+$$

$$= \text{man}(D' \cup \{d\}),$$

so $D' \cup \{d\}$ covers M^+ in P. Now if $|D'|$ were less than n, then $|D' \cup \{d\}| \leq n$, making $D' \cup \{d\}$ an explanation for M^+ in P. But since $d \in D' \cup \{d\}$, we would have $D' \cup \{d\} \in S_I$, contradicting $S_I = \varnothing$. Hence, $\text{order}(Q) = |D'| \geq n$. Combined with the result of the preceding paragraph, we therefore have $\text{order}(Q) = n$.

Theorem 2.7 (by induction on $N + |\text{scope}|$).

(1) *Base case*: $N = |scope| = 0$.

Case 1a: manifs $\neq \varnothing$. Then scope is not a cover for manifs, so R is not a diagnostic problem by Lemma 2.3. The test at line 3 of Genset succeeds, and the test at line 5 fails, so Genset terminates and returns $\varnothing$.

Case 1b: manifs $= \varnothing$. Then R is a diagnostic problem by Lemma 2.3, and as noted in Part I, $\text{Sol}(R) = \{\varnothing\}$. Thus, $\text{order}(R) = 0 = N$. The tests at lines 3 and 5 of Genset succeed, so it terminates and returns a generator set $\{\varnothing\}$. Note that $[\{\varnothing\}] = \{\varnothing\} = \text{Sol}(R)$.

(2) Induction step: Let $k > 0$, and suppose that the theorem hold whenever $N + |\text{scope}| < k$. Now let N and scope be such that $N + |\text{scope}| = k$.

Case 2a: R is a diagnostic problem of order N. Let $E \in \text{Sol}(R)$. Then by Lemma 1.6, the definition of prob, and Lemma 2.3, $E \subseteq \text{causes(manifs)} \subseteq \text{scope}$, so $|\text{scope}| \geq |E| = N$. Thus, the tests at lines 3 and 10 of Genset both fail, and lines 13–18 are executed. In line 13, $d \in \text{scope}$ is selected, and at line 14, $I = \{d' \in \text{scope} | \text{man}_p(d') \cap \text{manifs} = \text{man}_p(d) \cap \text{manifs}\}$. By Lemma 2.4, it immediately follows that $I = \{d' \in \text{scope} | \text{man}_R^+(d') = \text{man}_R^+(d)\}$. Let $S_I = \{E \in \text{Sol}(R) | E \cap I \neq \varnothing\}$, and $S_I' = \{E \in \text{Sol}(R) | E \cap I = \varnothing\}$. Since $S_I \cup S_I' = \text{Sol}(R) \neq \varnothing$ by [41, Lemma 1.3], S_I and S_I' cannot both be empty.

Case 2a1: $S_I = \varnothing$ and $S_I' \neq \varnothing$. Then $\text{Sol}(R) = S_I'$. By Lemma 2.5, prob(scope-I, manifs) is a diagnostic problem of order N, and Sol(prob(scope-I, manifs)) $= S_I'$. But $I \neq \varnothing$ since $d \in I$, so $N + |\text{scope-}I| < N + |\text{scope}| = k$. Thus, by the induction hypothesis, line 15 of Genset assigns F to be a generator set for S_I'. Similarly, by Lemma 2.6, prob(scope-I, manifs-man(d)) is a diagnostic problem of order N, so by the induction hypothesis H is assigned the value $\varnothing$ on line 16. Hence, G is assigned $\varnothing$ on line 17, and line 18 returns $F \cup G = F$ and terminates. Note that $[F] = S_I' = \text{Sol}(R)$.

Case 2a2: $S_I \neq \varnothing$ and $S_I' = \varnothing$. Then $\text{Sol}(R) = S_I$. By Lemma 2.5, either prob(scope-I, manifs) is not a diagnostic problem, or order(prob(scope-I, manifs)) $> N$, so by the induction hypothesis F is assigned $\varnothing$ on line 15. By Lemma 2.6, prob(scope-I, manifs-man(d)) is a diagnostic problem of order N-1, so by the induction hypothesis H is assigned a generator set for Sol(prob(scope-I, manifs-man(d))) in line 16. Thus, by Lemma 2.6, G is assigned a generator set in line 17 such that $[G] = S_I = \text{Sol}(R)$. Then $G \cup F = G$ is returned in line 18 and Genset terminates.

Case 2a3: $S_I \neq \varnothing$ and $S_I' \neq \varnothing$. Then as in case 2a1, F is set to a generator set for S_I', and as in case 2a2, G is set to a generator set for S_I. Therefore in line 18, Genset returns a generator set for $S_I \cup S_I' = \text{Sol}(R)$ and terminates.

Case 2b: R is not a diagnostic problem. Then by Lemma 2.3, scope is not a cover for manifs, so scope-I is not a cover for either manifs or manifs-man(d). Thus, from Lemma 2.3, both prob(scope-I, manifs) and prob(scope-I, manifs-man(d)) are not diagnostic problems. If |scope| $< N$, then Genset will return $\varnothing$ and terminate at line 11. Otherwise, from the induction hypothesis, F and H will be set to $\varnothing$ in lines 15 and 16, respectively, whence Genset will return $\varnothing$ and terminate at line 18.

Case 2*c*: *R is a diagnostic problem but* order(R) $> N$. By reasoning similar to the beginning of case 2a, lines 13–18 of Genset will be executed. Let d, I, S_I and S_I' be as defined in case 2a. If $S_I' = \varnothing$, then by Lemma 2.5 either prob(scope-I, manifs) is not a diagnostic problem, or order(prob(scope-I, manifs)) $> N+1$. If $S_I' \neq \varnothing$, then order(prob(scope-I, manifs)) $> N$. In either case, it follows from the induction hypothesis that F will be assigned $\varnothing$ in line 15. From Lemma 2.6, if $S_I \neq \varnothing$ then prob(scope-I, manifs-man(d)) is a diagnostic problem with order N or greater, and if $S_I = \varnothing$ then prob(scope-I, manifs-man(d)) is a diagnostic problem of order greater than N. In either case, it follows from the induction hypothesis that H will be set to $\varnothing$ in line 16, and thus G will be set to $\varnothing$ in line 17. Hence, Genset returns $\varnothing$ and terminates at line 18.

Theorem 2.8. Let $P = \langle \mathbf{D}_P, \mathbf{M}_P, \mathbf{C}_P, M^+ \rangle$ and let $k = \text{order}(P)$. Note that $P = \text{prob}(\mathbf{D}_P, M^+)$. Let $R = \text{prob}(\text{causes}_P(M^+), M^+) = \langle \mathbf{D}_R, \mathbf{M}_R, \mathbf{C}_R, M^+ \rangle$, where $\mathbf{D}_R = \text{causes}_p(M^+)$, $\mathbf{M}_R = \text{man}_p(\mathbf{D}_R)$, and $\mathbf{C}_R = \mathbf{C}_P \cap (\mathbf{D}_R \times \mathbf{M}_R)$. Let $D' = \mathbf{D}_P\text{-causes}(M^+)$.

Case 1: $D' = \varnothing$. Then $R = P$, so R is a diagnostic problem of order k and $Sol(R) = Sol(P)$.

Case 2: $D' \neq \varnothing$. Note that $\forall d' \in D'$, $\text{man}_P^+(d) = \varnothing$, and $\forall d'' \in \text{causes}_P(M^+)$, $\text{man}_P^+(d'') \neq \varnothing$. Select any $d \in D'$, and let $I = \{d' \in \mathbf{D}_P | \text{man}_P^+(d') = \text{man}_P^+(d)\}$. Then $I = \{d' \in D_p | \text{man}_p^+(d') = \varnothing\} = D'$. Also note that for any $E \in \text{Sol}(P)$, $E \subseteq \text{causes}_P(M^+)$ by Lemma 1.6, so $E \cap I = E \cap D' = \varnothing$. Thus if $S_I' = \{E \in \text{Sol}(P) | E \cap I = \varnothing\}$, then $S_I' = \text{Sol}(P)$. Thus, by Lemma 2.5(a), R is a diagnostic problem of order k and $\text{Sol}(R) = \text{Sol}(P)$.

In either case, $\text{Sol}(R) = \text{Sol}(P)$, so it suffices to prove that Solve[P] returns a generator set for Sol(R) and terminates.

When Solve[P] is called, n is initialized to zero, so $n \leq \text{order}(P) = \text{order}(R) = k$. Since $s = \varnothing$ initially, the test on line 5 is successful. If $n = k = 0$, then s will be assigned a nonempty generator set for Sol(R) at line 7 by Theorem 2.7, and the **while** test on line 5 will subsequently fail. Thus, line 10 will return s and Solve will terminate. If $n < k$, then by Theorem 2.7, Genset[causes(M^+), M^+, n] will terminate and return $\varnothing$, so n will be incremented and the **while** test will succeed again, as $s = \varnothing$. This will continue to occur until $n = k$, which must eventually happen because k is finite by Part I, Lemma 1.4. When $n = k$, then by Theorem 2.7, Genset[causes(M^+), M^+, k] will terminate and assign a nonempty generator set for Sol(R) as the value of s. Hence, the **while** loop will terminate at line 10 after returning $s = \text{Sol}(R) = \text{Sol}(P)$.

Lemma 2.9. Q is a diagnostic problem by definition, since P is a diagnostic problem and $M_Q^+ \subseteq \mathbf{M}$. There are two cases:

Case 1: There exists $E \in \mathrm{Sol}(P)$ such that $M^+ \cup \{m\} \subseteq \mathrm{man}(E)$. Then E is a minimal cover of $M^+ \cup \{m\}$, because if not (since it covers M^+) it would contradict the fact that E was an explanation in P. Hence, order$(Q) = n$.

Case 2: There does not exist $E \in \mathrm{Sol}(P)$ such that $M^+ \cup \{m\} \subseteq \mathrm{man}(E)$. Select any $E \in \mathrm{Sol}(P)$. Since E covers M^+ but not $M^+ \cup \{m\}$, it must be that $m \notin \mathrm{man}(E)$, so $\{m\} \cap \mathrm{man}(E) = \varnothing$. By Part I, Lemma 1.2, it follows that causes$(m) \cap E = \varnothing$. Select any $d \in$ causes(m), and let $D = E \cup \{d\}$. By the preceding, $d \notin E$, so $|D| = |E| + |\{d\}| = n + 1$. Furthermore, $M^+ \cup \{m\} \subseteq \mathrm{man}(E) \cup \mathrm{man}(d) = \mathrm{man}(D)$, so D covers $M^+ \cup \{m\}$. Suppose D were not a minimal cover of $M^+ \cup \{m\}$. Then there would be a D' that covered $M^+ \cup \{m\}$ with $|D'| \leq n$. But since D' covers M^+, $|D'| \geq$ order$(P) = n$. Thus, $|D'| = n$, so $D' \in \mathrm{Sol}(P)$, contradicting the assumption of case 2. Therefore, D must be a minimal cover for $M^+ \cup \{m\}$, so order$(Q) = |D'| = n + 1$.

Lemma 2.10. Since causes$(m) \neq \varnothing$ by Part I, Lemma 1.1(a′), $G/\mathrm{causes}(m)$ is defined, and H is a generator set by Lemma 1.19. By Lemma 1.20, $[H] = \{E \in [G] | E \cap \mathrm{causes}(m) \neq \varnothing\}$. Let $E' \in [H]$. Then $E' \in [G]$, and there exists $d \in E'$ such that $m \in \mathrm{man}(d) \subseteq \mathrm{man}(E')$, so E' covers m. Hence, $[H] \subseteq \{E \in [G] | E$ covers $m\}$. Similarly, let $E' \in \{E \in [G]/E \in$ covers $m\}$. Then there is a $d \in E'$ such that $m \in \mathrm{man}(d)$, so by Part I, Lemma 1.1(e), $d \in \mathrm{causes}(m)$. Hence, $E' \cap \mathrm{causes}(m) \neq \varnothing$, so $E' \in [H]$, and $\{E \in [G] | E$ covers $m\} \subseteq [H]$. Thus $[H] = \{E \in [G] | E$ covers $m\}$.

Lemma 2.11. Q is a diagnostic problem by definition, since P is a diagnostic problem and $M_Q^+ \subseteq \mathbf{M}$.

(a): Since $H = \varnothing$, $[H] = \varnothing$. But then by Lemma 2.10, $[H] = \{E \in [G] | E$ covers $m\} = \{E \in [G] | E$ covers $M^+ \cup \{m\}\} = \varnothing$. Thus there can be no covers of $M^+ \cup \{m\}$ of cardinality n, so order$(Q) \neq n$, so by Lemma 2.9, order$(Q) = n + 1$.

(b): Since $H \neq \varnothing$, it follows from Lemma 2.10 that $[H] = \{E \in [G] | E$ covers $m\} = \{E \in [G] | E$ covers $M^+ \cup \{m\}\} \neq \varnothing$, so order$(Q) = n$. For every $E' \in [H]$, $|E'| = n$ and E' covers $M^+ \cup \{m\}$ in Q, so $E' \in \mathrm{Sol}(Q)$. Thus $[H] \subseteq \mathrm{Sol}(Q)$. For every $E'' \in \mathrm{Sol}(Q)$, E'' covers $M^+ \cup \{m\}$, so E'' covers M^+ in P. Thus since $|E''| = n$, E'' is an explanation in P, so $E'' \in \mathrm{Sol}(P) = [G]$. Hence, $E'' \in \{E \in [G] | E$ covers $m\} = [H]$. Thus, $\mathrm{Sol}(Q) \subseteq [H]$, so by the above, $[H] = \mathrm{Sol}(Q)$.

Lemma 2.12. (a),(b): Immediate from induction and the definition of manifs$_i$ and scope$_i$.

(c): Immediate by induction using Lemma 2.3.

Theorem 2.13 (by induction of i).

Case 1 (Base case). $i = 0$. Then $P_0 = \text{prob}(\varnothing, \varnothing) = \langle \varnothing, \varnothing, \varnothing, \varnothing \rangle$, so $\text{order}(P_0) = 0 = n_0$. Also, $\text{Sol}(P_0) = \{\varnothing\} = [\{\varnothing\}] = [\text{hypothesis}_0]$.

Case 2 (Induction step). Let $I > 0$, and suppose the theorem holds for $i < I$. Consider the case where $i = I$. From the induction hypothesis, we know that $n_{i-1} = \text{order}(P_{i-1})$ and $[\text{hypothesis}_{i-1}] = \text{Sol}(P_{i-1})$. From the definition of Nextman we know $m_i \notin \text{manifs}_i$. Thus, by Lemma 2.9 there are two cases.

Case 2a: $\text{order}(P_i) = \text{order}(P_{i-1}) = n_{i-1}$. Since $\text{order}(P_i) \neq n_{i-1} + 1$, then by Lemma 2.11(a) (contrapositive), $\text{hypothesis}_{i-1}/\text{causes}(m_i) \neq \varnothing$. Thus, the test on line 13 of HT fails and lines 14–16 are not executed, so $n_i = n_{i-1}$ and $\text{hypothesis}_i = \text{hypothesis}_{i-1}/\text{causes}(m_i)$. It therefore follows from Lemma 2.11(b) that $n_i = \text{order}(P_i)$, and $[\text{hypothesis}_i] = \text{Sol}(P_i)$.
Case 2b: $\text{order}(P_i) = \text{order}(P_{i-1}) + 1$. Then, since $\text{order}(P_i) \neq n_{i-1}$, by Lemma 2.11(b) (contrapositive), $\text{hypothesis}_{i-1}/\text{causes}(m_i) = \varnothing$ at line 13 of HT, so lines 14–16 are executed. Line 14 assigns $n_i = n_{i-1} + 1 = \text{order}(P_{i-1}) + 1 = \text{order}(P_i)$. By Theorem 2.7, the call to $\text{Genset}[\text{scope}_i, \text{manifs}_i, n_i]$ assigns hypothesis_i to a generator set for $\text{Sol}(P_i)$.

Corollary 2.14. By the definition of Moremanifs, the **while** loop in HT is executed $|M^+|$ times. Thus, by Theorem 2.13, HT returns $\text{hypothesis}_{|M^+|}$ at line 18, a generator set for $\text{Sol}(R)$, where $R = P_{|M^+|} = \text{prob}(\text{scope}_{|M^+|}, \text{manifs}_{|M^+|}) = \text{prob}(\text{causes}(M^+), M^+)$. By reasoning similar to that in the proof of Theorem 2.8 (first four paragraphs), it follows that $\text{Sol}(P) = \text{Sol}(R)$, so HT returns a generator set for $\text{Sol}(P)$ at line 18 and terminates.

Proposition 2.15. Proof by induction on $N = |M^+|$.

Base case: $N = 1$. Then $M^+ = \{m\}$ and the sequence of steps produces $G = a$ division of $\{(\mathbf{D})\}$ by *causes*(m). By Part *I*, Lemma 1.19, G is a generator set. Let $D' = \mathit{causes}(m) \neq \varnothing$. By the definition of $\mathbf{C}$ and the division operation, $G = \{(\mathbf{D} \cap \mathit{causes}(m))\} = \{(\mathit{causes}(m))\} = \{(D')\}$. Here, $D' = \bigcap_{m \in M^+} \mathit{causes}(m)$ trivially.
Induction step. Assume that the proposition is true for all subsets of M^+ of cardinality $N - 1$, and let M^+ have cardinality N. Let m' be the last member of M^+ to be chosen in the **for** loop, and let $M = M^+ - \{m'\}$. Then $|M| = N - 1$. By the inductive hypothesis, the first $N - 1$ steps of the **for** loop give a generator set $\{(D'')\}$, where $D'' = \bigcap_{m \in M} \text{causes}(m)$ and $D'' \neq \varnothing$. Then by the reasoning immediately above the definition of the division operation, on the last pass through the **for** loop G is assigned $\{(D'' \cap \text{causes}(m'))\} = \{(D')\}$, where $D' = \bigcap_{m \in M^+} \text{causes}(m)$. Note that this last assertion makes implicit use of the single-disorder assumption.

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