AMSC/CMSC 460 Quiz 3 , Fall 2002

1.(10) Suppose that we have a function f defined on the interval $[0, 2]$, and we want to approximate it by a piecewise linear function that is never further than 10^{-4} from f. If the second derivative of f is bounded by 36, how many equally-spaced points should be use?

Answer: (Change "bounded" to "bounded in absolute value".) If p is the piecewise linear function, and $x \in [x_i, x_{i+1}]$, the error is

$$|f(x) - p(x)| = |\frac{(x - x_i)(x - x_{i+1})}{2} f"(\xi)|$$

where ξ is some point in $[x_i, x_{i+1}]$. Now, $|(x - x_i)(x - x_{i+1})| \leq h^2$, with $h = x_{i+1} - x_i$. Therefore, the error is bounded by

$$|f(x) - p(x)| \leq \frac{h^2}{2} 36 \, .$$

We want this quantity to be less than 10^{-4}, so we need

$$h \leq \sqrt{\frac{10^{-4}}{18}} \, .$$

Since $h = (2 - 0)/(n - 1)$, where the number of points is n, we have

$$n \geq 850.$$

(In fact, $|(x - x_i)(x - x_{i+1})| \leq (h/2)^2$, so a smaller number of points, 425, is sufficient.)

2.(10) Recall the basis that we are using for cubic splines with knots $x_1 < x_2 < \ldots < x_n$: We will set $s(x)$ equal to $s_{i+1}(x)$ on interval I_{i+1}, where

$$s_{i+1}(x) = m_i \frac{(x_{i+1} - x)^3}{6h_{i+1}} + m_{i+1} \frac{(x - x_i)^3}{6h_{i+1}} + a_i(x - x_i) + b_i$$

for some constants m_i, m_{i+1}, a_i, and b_i, where

- $h_{i+1} = x_{i+1} - x_i$, $i = 1, \ldots, n-1$
- $k_{i+1} = f_{i+1} - f_i$, $i = 1, \ldots, n-1$
- $I_{i+1} = [x_i, x_{i+1}]$, $i = 1, \ldots, n-1$

Write down the conditions that guarantee that s'' is continuous at the knots $x_2, \ldots x_{n-1}$.

Answer: This problem was more confusing than I intended. Assuming that $x_1 \leq x_2 \leq \ldots \leq x_n$, we have

$$s'_{i+1}(x) = -\frac{m_i}{2h_{i+1}}(x_{i+1} - x)^2 + \frac{m_{i+1}}{2h_{i+1}}(x - x_i)^2 + a_i \, .$$

$$s''_{i+1}(x) = +\frac{m_i}{h_{i+1}}(x_{i+1} - x) + \frac{m_{i+1}}{h_{i+1}}(x - x_i) \, .$$

Now, evaluating $s(x_i)$, $i = 2, \ldots, n-1$, we obtain

$$s''_{i+1}(x_i) = m_i = s''_i(x_i)$$

so continuity is guaranteed.