

AMSC/CMSC 460      Final Exam      ,      Fall 2007

Show all work. You may leave arithmetic expressions in any form that a calculator could evaluate. By putting your name on this paper, you agree to abide by the university's code of academic integrity in completing the quiz. Use no books, calculators, cellphones, communication with others, scratchpaper, etc. You may use one sheet of paper, in your own handwriting, with notes about the course material.

Name \_\_\_\_\_

Student number \_\_\_\_\_

Problem 1: (20) \_\_\_\_\_

Problem 2: (25) \_\_\_\_\_

Problem 3: (20) \_\_\_\_\_

Problem 4: (20) \_\_\_\_\_

Problem 5: (20) \_\_\_\_\_

Problem 6: (25) \_\_\_\_\_

Problem 7: (20) \_\_\_\_\_

Total: (150) \_\_\_\_\_

1. Mark each of the following statements T (True) or F (False).

Grading: 2 \* (number correct) - 1.5 \* (number incorrect) + 0 \* (number blank).

- a. \_\_\_\_\_ If  $A$  and  $B$  are  $1000 \times 1000$  matrices and  $c$  is a  $1000 \times 1$  vector, then  $A * (B * c)$  is faster in Matlab than  $(A * B) * c$ .
- b. \_\_\_\_\_ The formula  $y_{n+1} = y_n + \frac{h_n}{2}(f_{n+1} + f_n)$  represents an implicit method for solving ODE's.
- c. \_\_\_\_\_ Matlab stores matrices by row.
- d. \_\_\_\_\_ Suppose  $A$  is a  $1000 \times 1000$  nonsingular matrix with no zero elements. It takes fewer floating point operations to form  $A^{-1}$  and solve a linear system by multiplying  $A^{-1}b$  than to factor the matrix  $A = LU$  and use forward and backward substitution to solve the system.
- e. \_\_\_\_\_ Gaussian integration uses equally spaced points.
- f. \_\_\_\_\_ When adding numbers, relative error bounds add.
- g. \_\_\_\_\_ The software `ode45` should not be used for stiff systems of differential equations.
- h. \_\_\_\_\_ Adaptive numerical integration routines use more points in regions where the function is changing rapidly.
- i. \_\_\_\_\_ The spacing between floating point numbers is uniform.
- j. \_\_\_\_\_ The  $\infty$ -norm of the matrix  $A = [ 3, -6; 5 1]$  is 9.

2. Suppose we have an approximate solution  $\hat{\mathbf{x}}$  to the linear system of equations  $\mathbf{Ax} = \mathbf{b}$ . Suppose  $\|\mathbf{A}\| = 10$ ,  $\|\mathbf{A}^{-1}\| = 20$ ,  $\|\mathbf{b}\| = 45$ ,  $\|\hat{\mathbf{x}}\| = 5$ , and  $\|\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}\| = 10^{-3}$ .

a. (10) Give a bound on the forward error. (You may use either relative or absolute error.)

b. (15) Give a bound on the backward error. (You may use either relative or absolute error.)

3. Given  $n$  numbers  $a_1, \dots, a_n$  and a number  $x$ , consider the following algorithm:

```
 $p = a_1$   
For  $j = 2, \dots, n$ ,  
     $p = p * x + a_j$   
end for
```

a. (10) How many multiplications does the algorithm perform?

b. (10) What does the algorithm compute?

4.

a. (10) When we say that the error formula for an Adams method is  $\frac{5h^3}{12}y^{(3)}(\eta)$ , what do we mean? (Define all notation and give an unambiguous statement about what error you mean and what the error formula says.)

b. (10) Suppose we use an Adams PECE scheme to solve a differential equation  $y' = f(t, y)$  and obtain  $y_{n+1}^P = 1.2450$  and  $y_{n+1}^C = 1.2420$ . Suppose the error formula for the predictor is  $\frac{5h^3}{12}y^{(3)}(\eta)$  and for the corrector is  $\frac{h^4}{24}y^{(4)}(\xi)$ . What can you say about the error?

5. (20) Suppose we have data points  $(t_i, y_i)$ ,  $i = 1, \dots, 20$  and we believe that

$$p(t) = x_1 e^{tx_2} + x_3 e^{tx_4}$$

is a good model to  $y$  as a function of  $t$ . We decide to solve a least squares problem to fit the parameters  $x_1, \dots, x_4$ . (The model is nonlinear in the parameters, so we can't use backslash.)

Write a Matlab function that takes the vector  $\mathbf{x}$  as input and computes the 2-norm of the vector that is the difference between the data values  $y_i$  and the values predicted by the model.

6.

a. (15) Suppose that  $\mathbf{A}$  is a  $5 \times 2$  matrix with rank 2. Let  $[\mathbf{Q}, \mathbf{R}] = \text{qr}(\mathbf{A})$  be the QR decomposition of  $\mathbf{A}$ . (In other words,  $\mathbf{A} = \mathbf{QR}$ .) Using this decomposition, we can find a matrix  $\mathbf{Z}$  so that  $\{\mathbf{x} : \mathbf{A}^T \mathbf{x} = \mathbf{0}\} = \{\mathbf{Z}\mathbf{y} : \mathbf{y} \text{ is arbitrary}\}$ . What is the matrix  $\mathbf{Z}$ ?

b. (10) Suppose that `fmin` is a Matlab function that finds a (local unconstrained) minimizer of a given function `g.m`. Consider the optimization problem

$$\min_{\mathbf{x}} f(\mathbf{x})$$

subject to the constraint that  $\mathbf{A}^T \mathbf{x} = \mathbf{0}$ . Assuming that you have the matrix  $\mathbf{Z}$  from part a (even if you don't know how to compute it) and a function `f.m` that evaluates the function  $f$ , write a Matlab function `g.m` so that we can solve the optimization problem using `fmin`.

7. Let

$$I = \int_1^2 \sqrt{x} \, dx.$$

a. (10) Write the formula for the composite trapezoidal rule estimate to  $I$  using 5 panels (i.e.,  $h = 1/5$ ).

b. (10) How many panels are necessary to ensure that the (absolute) error is less than  $10^{-6}$ ? (Express the number of panels in a formula that Matlab could evaluate, recalling that the error formula for the composite trapezoidal rule is  $-(b-a)h^2 f''(\eta)/12$ .)

1. Mark each of the following statements T (True) or F (False).

Grading: 2 \* (number correct) - 1.5 \* (number incorrect) + 0 \* (number blank).

- a. T Sec5.2 If  $A$  and  $B$  are  $1000 \times 1000$  matrices and  $c$  is a  $1000 \times 1$  vector, then  $A * (B * c)$  is faster in Matlab than  $(A * B) * c$ .
- b. T p354 The formula  $y_{n+1} = y_n + \frac{h_n}{2}(f_{n+1} + f_n)$  represents an implicit method for solving ODE's.
- c. F Matlab stores matrices by row.
- d. F Sec6.3.5 Suppose  $A$  is a  $1000 \times 1000$  nonsingular matrix with no zero elements. It takes fewer floating point operations to form  $A^{-1}$  and solve a linear system by multiplying  $A^{-1}b$  than to factor the matrix  $A = LU$  and use forward and backward substitution to solve the system.
- e. F p155 Gaussian integration uses equally spaced points.
- f. F ErrorSlide 43 When adding numbers, relative error bounds add.
- g. T ode p18 The software `ode45` should not be used for stiff systems of differential equations.
- h. T int part 2 Adaptive numerical integration routines use more points in regions where the function is changing rapidly.
- i. F ErrorSlide 37 The spacing between floating point numbers is uniform.
- j. T p185 The  $\infty$ -norm of the matrix  $A = [ 3, -6; 5 1]$  is 9.

2. Suppose we have an approximate solution  $\hat{\mathbf{x}}$  to the linear system of equations  $\mathbf{Ax} = \mathbf{b}$ . Suppose  $\|\mathbf{A}\| = 10$ ,  $\|\mathbf{A}^{-1}\| = 20$ ,  $\|\mathbf{b}\| = 45$ ,  $\|\hat{\mathbf{x}}\| = 5$ , and  $\|\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}\| = 10^{-3}$ .

a. (10) Give a bound on the forward error. (You may use either relative or absolute error.)

**Answer:** (ErrorSlide 53-60 and Unit 4 Part 2 notes p10)

$$\frac{\|\mathbf{x}_{true} - \hat{\mathbf{x}}\|}{\|\mathbf{x}_{true}\|} \leq \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \kappa(\mathbf{A}) = \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \|\mathbf{A}\| \|\mathbf{A}^{-1}\|.$$

Therefore,

$$\frac{\|\mathbf{x}_{true} - \hat{\mathbf{x}}\|}{\|\mathbf{x}_{true}\|} \leq \frac{200 * 10^{-3}}{45}.$$

b. (15) Give a bound on the backward error. (You may use either relative or absolute error.)

**Answer:** (Error notes, p18)

We have exactly solved the linear system  $\mathbf{Ax} = \mathbf{b} - \mathbf{r}$  where  $\|\mathbf{r}\| = \|\mathbf{b} - \mathbf{Ax}\| = 10^{-3}$ .

3. Given  $n$  numbers  $a_1, \dots, a_n$  and a number  $x$ , consider the following algorithm:

```

p = a1
For j = 2, ..., n,
    p = p * x + aj
end for
    
```

a. (10) How many multiplications does the algorithm perform?

**Answer:**  $n - 1$ .

b. (10) What does the algorithm compute?

**Answer:** (Unit 3 notes p.8)

This is Horner's rule for evaluating the polynomial

$$p(x) = a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n.$$

4.

a. (10) When we say that the error formula for an Adams method is  $\frac{5h^3}{12}y^{(3)}(\eta)$ , what do we mean? (Define all notation and give an unambiguous statement about what error you mean and what the error formula says.)

**Answer:** (Quiz 6 answer)

We are solving the differential equation  $y' = f(t, y)$ . The error formula means that the difference between the true value and the value computed by the Adams formula, assuming that all of the function values used in the formula are correct, is  $\frac{5h^3}{12}y^{(3)}(\eta)$ , where  $h$  is the difference between the new  $t$  value and the old one and  $\eta$  is an unknown point between the two  $t$  values.

b. (10) Suppose we use an Adams PECE scheme to solve a differential equation  $y' = f(t, y)$  and obtain  $y_{n+1}^P = 1.2450$  and  $y_{n+1}^C = 1.2420$ . Suppose the error formula for the predictor is  $\frac{5h^3}{12}y^{(3)}(\eta)$  and for the corrector is  $\frac{h^4}{24}y^{(4)}(\xi)$ . What can you say about the error?

**Answer:** (Quiz 6 answer)

The error formulas tell us that

$$\begin{aligned}y_{n+1} - y_{n+1}^P &= \frac{5h^3}{12}y^{(3)}(\eta), \\y_{n+1} - y_{n+1}^C &= \frac{h^4}{24}y^{(4)}(\xi).\end{aligned}$$

We can't compute the left or right-hand sides, but if we subtract the two equations we obtain

$$y_{n+1}^C - y_{n+1}^P = \frac{5h^3}{12}y^{(3)}(\eta) - \frac{h^4}{24}y^{(4)}(\xi),$$

and the left-hand side is computable. Assuming that the error in the corrector is much smaller than the error in the predictor, we have

$$|y_{n+1} - y_{n+1}^P| \approx |y_{n+1}^C - y_{n+1}^P| = 1.2450 - 1.2420 = .003.$$

This is an approximation to the **local error** in the predictor, defined in part a.

5. (20) Suppose we have data points  $(t_i, y_i)$ ,  $i = 1, \dots, 20$  and we believe that

$$p(t) = x_1 e^{tx_2} + x_3 e^{tx_4}$$

is a good model to  $y$  as a function of  $t$ . We decide to solve a least squares problem to fit the parameters  $x_1, \dots, x_4$ . (The model is nonlinear in the parameters, so we can't use backslash.)

Write a Matlab function that takes the vector  $\mathbf{x}$  as input and computes the 2-norm of the vector that is the difference between the data values  $y_i$  and the values predicted by the model.

**Answer:** (Unit 5 notes, p1)

```
function z = f(x)
global t y
p = x(1) * exp(t * x(2)) + x(3) * exp(t * x(4));
% r is the difference between the data values and the predicted values.
r = y - p;
z = norm(r)
```

6.

a. (15) Suppose that  $\mathbf{A}$  is a  $5 \times 2$  matrix with rank 2. Let  $[\mathbf{Q}, \mathbf{R}] = \text{qr}(\mathbf{A})$  be the QR decomposition of  $\mathbf{A}$ . (In other words,  $\mathbf{A} = \mathbf{QR}$ .) Using this decomposition, we can find a matrix  $\mathbf{Z}$  so that  $\{\mathbf{x} : \mathbf{A}^T \mathbf{x} = \mathbf{0}\} = \{\mathbf{Z}\mathbf{y} : \mathbf{y} \text{ is arbitrary}\}$ . What is the matrix  $\mathbf{Z}$ ?

**Answer:** (Unit 7 notes, p5)

The matrix  $\mathbf{Z}$  is the last three columns of the matrix  $\mathbf{Q}$ :  $\mathbf{Z} = \mathbf{Q}(:, 3:5)$ .

b. (10) Suppose that `fmin` is a Matlab function that finds a (local unconstrained) minimizer of a given function `g.m`. Consider the optimization problem

$$\min_{\mathbf{x}} f(\mathbf{x})$$

subject to the constraint that  $\mathbf{A}^T \mathbf{x} = \mathbf{0}$ . Assuming that you have the matrix  $\mathbf{Z}$  from part a (even if you don't know how to compute it) and a function `f.m` that evaluates the function  $f$ , write a Matlab function `g.m` so that we can solve the optimization problem using `fmin`.

**Answer:** Function minimization notes, plus composite use of functions, like Quiz 7, problem 1.

Minimizing  $f(\mathbf{x})$  over all vectors  $\mathbf{x}$  satisfying  $\mathbf{A}^T \mathbf{x} = \mathbf{0}$  is the same as minimizing  $f(\mathbf{x})$  over all vectors  $\mathbf{x} = \mathbf{Z}\mathbf{y}$ . Therefore, we want to minimize  $f(\mathbf{Z}\mathbf{y})$  over all vectors  $\mathbf{y}$  ( $3 \times 1$ ). The function `g.m` needs to evaluate  $f(\mathbf{Z}\mathbf{y})$ , given  $\mathbf{y}$ .

```
function t = g(y)
global Z
x = Z * y;
t = f(x);
```

7. Let

$$I = \int_1^2 \sqrt{x} dx.$$

a. (10) Write the formula for the composite trapezoidal rule estimate to  $I$  using 5 panels (i.e.,  $h = 1/5$ ).

**Answer:** (Unit 3, part 1 notes p.6)

$$T = \frac{1}{5} \left( \frac{\sqrt{1}}{2} + \sqrt{1+h} + \sqrt{1+2h} + \sqrt{1+3h} + \sqrt{1+4h} + \frac{\sqrt{2}}{2} \right).$$

b. (10) How many panels are necessary to ensure that the (absolute) error is less than  $10^{-6}$ ? (Express the number of panels in a formula that Matlab could evaluate, recalling that the error formula for the composite trapezoidal rule is  $-(b-a)h^2 f''(\eta)/12$ .)

**Answer:** (Unit 3, part 1 notes p.6)

If  $f(x) = x^{1/2}$ , then  $|f''(x)| = |-x^{-3/2}/4|$  is bounded by  $1/4$  in the interval  $[1, 2]$ . Therefore, since  $b - a = 2 - 1 = 1$ , to make the error less than  $10^{-6}$ , it is sufficient to make

$$\frac{h^2}{4 * 12} < 10^{-6},$$

so

$$h < \sqrt{48 * 10^{-6}}.$$

The number of panels should be at least  $\lceil \frac{1}{\sqrt{48 * 10^{-6}}} \rceil$ .