1. (10) Compute the quadratic polynomial that interpolates the data

$$(x, f(x)) = (0,5), \ \ (1,11), \ \ (2,21).$$

Use either the Lagrange form or the Newton form.

Answer:

The Lagrange form is

$$p(x) = 5\frac{(x-1)(x-2)}{(0-1)(0-2)} + 11\frac{x(x-2)}{(1)(1-2)} + 21\frac{x(x-1)}{(2)(2-1)}.$$

The divided differences are:

5		
11	6	
21	10	(6-10)/(0-2) = 2

so the Newton form is

$$p(x) = 5 + 6x + 2x(x-1).$$

2. (10) Suppose we are interested in approximating a function $f(x)$ on the interval $[-1, 1]$ using a polynomial p_{n-1} that interpolates f at the n points given by $-1, -1+h, -1+2h, \ldots, 1$, where $h = 2/(n-1)$. Suppose you know that, on this interval, the maximum absolute value of all derivatives of f is 25:

$$\max_{x \in [-1,1]} |f^{(k)}(x)| < 25, \quad k = 0, 1, \ldots.$$

Describe how you would determine how many interpolation points you should use to guarantee that

$$|f(x) - p_{n-1}(x)| \leq 10^{-3}, \quad \text{for all} \quad x \in [-1, 1].$$

In particular, write a sequence of Matlab statements to verify that a particular value of n was large enough.

A useful formula:

$$f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi)}{n!}(x - x_1) \ldots (x - x_n).$$

Answer: We need to verify that the absolute value of the right hand side of the useful formula is bounded by 10^{-3}. Since $\xi \in [-1, 1]$, we have the bound of 25 for the derivative. We know $x_i = -1 + ih$ where $h = 2/(n-1)$ is the distance between the n equally spaced points. We need a bound on $g(x) = (x - x_1) \ldots (x - x_n)$.

One bound is as follows. We saw graphically that the max occurs when x is in one of the end intervals. Let's put it in the first interval. Then

$$\begin{aligned}
|x - x_1| &\leq h/2, \\
|x - x_2| &\leq 2h, \\
|x - x_3| &\leq 3h, \\
&\ldots \\
|x - x_n| &\leq nh.
\end{aligned}$$

(The computation for x in the last interval is similar.) So for $x \in [-1, 1]$,

$$|g(x)| \leq \frac{n!}{2} h^n.$$

Therefore, our error bound of 10^{-3} is satisfied if

$$\frac{25}{2}\left(\frac{2}{n-1}\right)^{n} \leq 10^{-3}.$$

We could solve this expression for n (by taking logs of both sides) or let Matlab evaluate it to see whether a given n is large enough.