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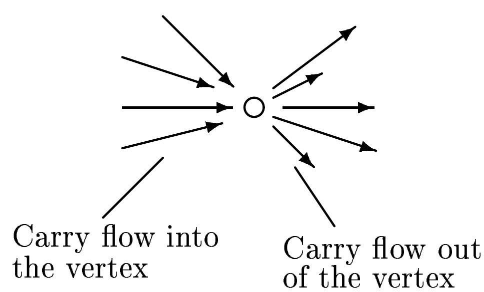
15 Network Flow - Maximum Flow Problem

Read [21, 5, 19].

The problem is defined as follows: Given a directed graph $G^d = (V, E, s, t, c)$ where s and t are special vertices called the source and the sink, and c is a capacity function $c : E \to \Re^+$, find the maximum flow from s to t.

Flow is a function $f : E \to \Re$ that has the following properties:

1. **(Skew Symmetry)** $f(u, v) = -f(v, u)$
2. **(Flow Conservation)** $\Sigma_{v \in V} f(u, v) = 0$ for all $u \in V - \{s, t\}$.
   (Incoming flow) $\Sigma_{v \in V} f(v, u) =$ (Outgoing flow) $\Sigma_{v \in V} f(u, v)$



Flow Conservation

3. **(Capacity Constraint)** $f(u, v) \le c(u, v)$

Maximum flow is the maximum value $|f|$ given by

$$|f| = \Sigma_{v \in V} f(s, v) = \Sigma_{v \in V} f(v, t).$$

Definition 15.1 (Residual Graph) *G_f^D is defined with respect to some flow function f, $G_f = (V, E_f, s, t, c')$ where $c'(u, v) = c(u, v) - f(u, v)$. Delete edges for which $c'(u, v) = 0$.*

As an example, if there is an edge in G from u to v with capacity 15 and flow 6, then in G_f there is an edge from u to v with capacity 9 (which means, I can still push 9 more units of liquid) and an edge from v to u with capacity 6 (which means, I can cancel all or part of the 6 units of liquid I'm currently pushing) [1]. E_f contains all the edges e such that $c'(e) > 0$.

Lemma 15.2 *Here are some easy to prove facts:*

1. *f' is a flow in G_f iff $f + f'$ is a flow in G.*
2. *f' is a maximum flow in G_f iff $f + f'$ is a maximum flow in G.*
3. *$|f + f'| = |f| + |f'|$.*

[1] Since there was no edge from v to u in G, then its capacity was 0 and the flow on it was -6. Then, the capacity of this edge in G_f is $0 - (-6) = 6$.

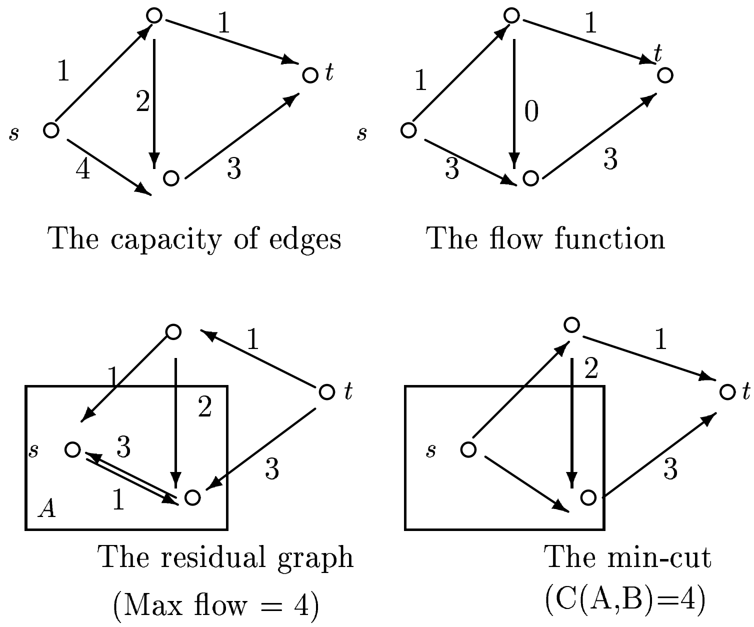


Figure 14: An example

4. If f is a flow in G, and f^* is the maximum flow in G, then $f^* - f$ is the maximum flow in G_f.

Definition 15.3 (Augmenting Path) *A path P from s to t in the residual graph G_f is called augmenting if for all edges (u, v) on P, $c'(u, v) > 0$. The* residual capacity *of an augmenting path P is $\min_{e \in P} c'(e)$.*

The idea behind this definition is that we can send a positive amount of flow along the augmenting path from s to t and "augment" the flow in G. (This flow increases the real flow on some edges and cancels flow on other edges, by reversing flow.)

Definition 15.4 (Cut) *An (s, t) cut is a partitioning of V into two sets A and B such that $A \cap B = \emptyset$, $A \cup B = V$ and $s \in A, t \in B$.*

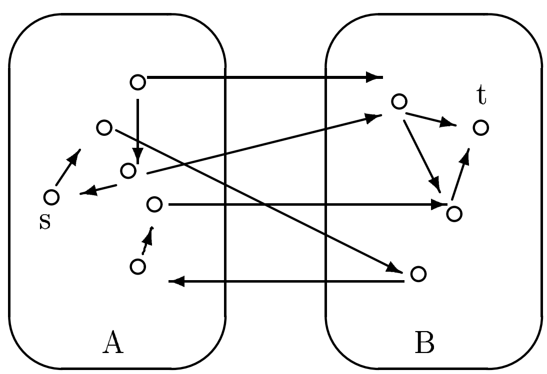


Figure 15: An (s, t) Cut

Definition 15.5 (Capacity Of A Cut) *The capacity $C(A, B)$ is given by*

$$C(A, B) = \Sigma_{a \in A, b \in B} \; c(a, b).$$

By the capacity constraint we have that $|f| = \Sigma_{v \in V} f(s,v) \leq C(A,B)$ for any (s,t) cut (A,B). Thus the capacity of the minimum capacity s,t cut is an upper bound on the value of the maximum flow.

Theorem 15.6 (Max flow - Min cut Theorem) *The following three statements are equivalent:*

1. *f is a maximum flow.*
2. *There exists an (s,t) cut (A,B) with $C(A,B) = |f|$.*
3. *There are no augmenting paths in G_f.*

An augmenting path is a directed path from s to t.

Proof:
We will prove that (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

((2) $\Rightarrow$ (1)) Since no flow can exceed the capacity of an (s,t) cut (i.e. $f(A,B) \leq C(A,B)$), the flow that satisfies the equality condition of **(2)** must be the maximum flow.

((1) $\Rightarrow$ (3)) If there was an augmenting path, then I could augment the flow and find a larger flow, hence f wouldn't be maximum.

((3) $\Rightarrow$ (2)) Consider the residual graph G_f. There is no directed path from s to t in G_f, since if there was this would be an augmenting path. Let $A = \{v | v$ is reachable from s in $G_f\}$. A and $\overline{A}$ form an (s,t) cut, where all the edges go from $\overline{A}$ to A. The flow f' must be equal to the capacity of the edge, since for all $u \in A$ and $v \in \overline{A}$, the capacity of (u,v) is 0 in G_f and $0 = c(u,v) - f'(u,v)$, therefore $c(u,v) = f'(u,v)$. Then, the capacity of the cut in the original graph is the total capacity of the edges from A to $\overline{A}$, and the flow is exactly equal to this amount. $\square$

A "Naive" Max Flow Algorithm:

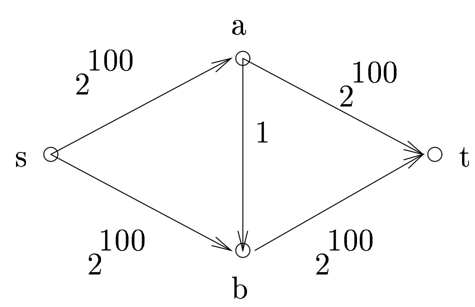
Initially let f be the 0 flow
while (there is an augmenting path P in G_f) **do**
    $c(P) \leftarrow \min_{e \in P} c'(e)$;
    send flow amount $c(P)$ along P;
    update flow value $|f| \leftarrow |f| + c(P)$;
    compute the new residual flow network G_f

Analysis: The algorithm starts with the zero flow, and stops when there are no augmenting paths from s to t. If all edge capacities are integral, the algorithm will send at least one unit of flow in each iteration (since we only retain those edges for which $c'(e) > 0$). Hence the running time will be $O(m|f^*|)$ in the worst case ($|f^*|$ is the value of the max-flow).

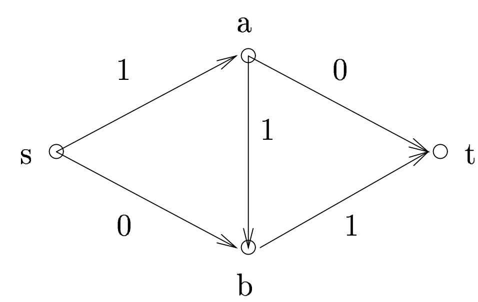
A worst case example. Consider a flow graph as shown on the Fig. 16. Using augmenting paths (s,a,b,t) and (s,b,a,t) alternatively at odd and even iterations respectively (fig.1(b-c)), it requires total $|f^*|$ iterations to construct the max flow.

If all capacities are rational, there are examples for which the flow algorithm might never terminate. The example itself is intricate, but this is a fact worth knowing.

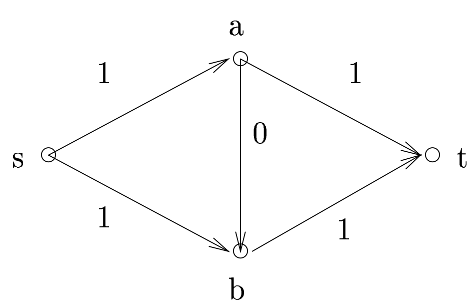
Example. Consider the graph on Fig. 17 where all edges except (a,d), (b,e) and (c,f) are unbounded (have comparatively large capacities) and c$(a,d) = 1$, c$(b,e) = R$ and c$(c,f) = R^2$. Value R is chosen such that $R = \frac{\sqrt{5}-1}{2}$ and, clearly (for any $n \geq 0$, $R^{n+2} = R^n - R^{n+1}$. If augmenting paths are selected as shown on Fig. 17 by dotted lines, residual capacities of the edges (a,d), (b,e) and (c,f) will remain 0, R^{3k+1} and R^{3k+2} after every $(3k+1)$st iteration ($k = 0,1,2,\ldots$). Thus, the algorithm will never terminate.



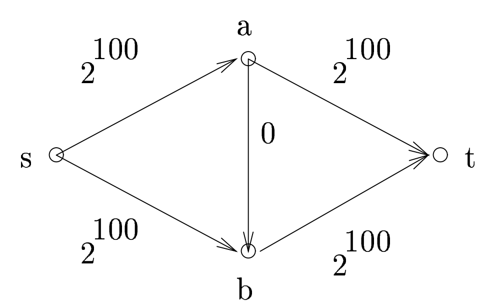
(a) Initial flow graph with capacities



(b) Flow in the graph after the 1st iteration



(c) Flow in the graph after the 2nd iteration



(d) Flow in the graph after the final iteration

Figure 16: Worst Case Example

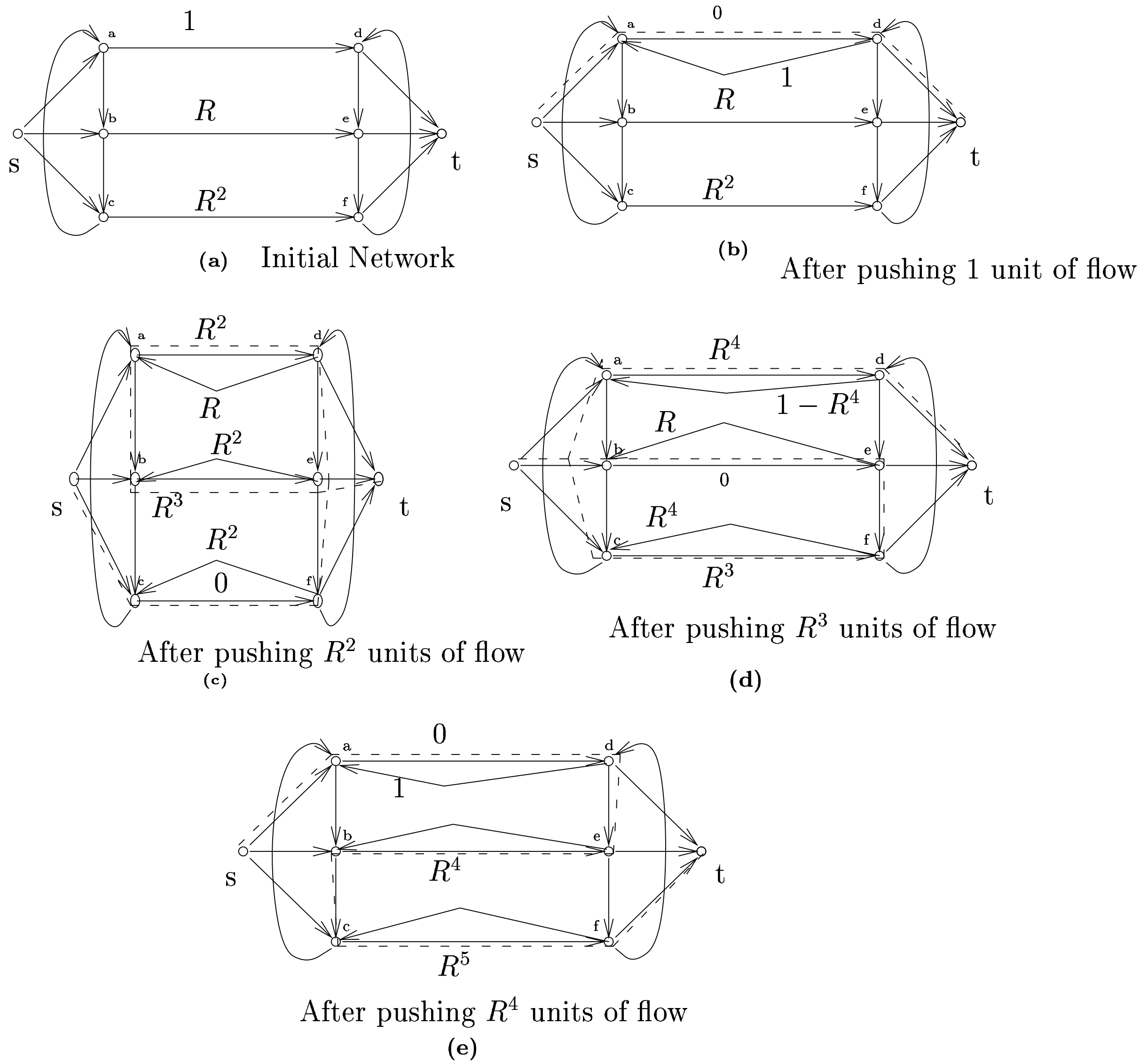


Figure 17: Non-terminating Example