Notes by Samir Khuller

9 Nemhauser-Trotter Theorem

Consider the IP formulation for the weighted vertex cover problem. When we relax this to a linear program and solve it, it turns out that any extreme point solution (there always exists an optimal solution extreme point solution) has the property that the variables take one of three possible values $\{0, \frac{1}{2}, 1\}$. There are various ways of proving this theorem. However, our primary concern is of showing the following interesting theorem originally proven by Nemhauser and Trotter. The proof given here is simpler than their proof.

First some defintions.

1. V_0 is the set of nodes whose fractional values are 0.
2. V_1 is the set of nodes whose fractional values are 1.
3. $V_{\frac{1}{2}}$ ais the set of nodes whose fractional values are 1/2.

Theorem 9.1 *(Nemhauser and Trotter) There exists an optimal solution OPT with the following properties*

(a) OPT is a subset of $(V_1 \cup V_{\frac{1}{2}})$.

(b) OPT includes all of V_1.

Proof:

We will only prove (a). Note that if OPT satisfies (a) then it must also satisfy (b). Let us see why. Suppose OPT does not include $v \in V_1$. If v does not have any neighbors in V_0 then we can reduce the fractional variable X_v and get a BETTER LP solution. This assumes that the weight of the node is > 0. If the weight of the node is 0, then clearly we can include it in OPT without increasing the weight of OPT. If v has a neighbor in V_0, then OPT does not cover this edge!

We will now prove (a). Suppose OPT picks nodes from V_0. Let us call $(OPT \cap V_0)$ as B_I (bad guys in) and $(V_1 \setminus OPT)$ as L_O (left out guys). The nodes in B_I can have edges to ONLY nodes in V_1. The edges going to L_O are the only ones that have one end chosen, all the other edges are double covered. The nodes in L_O have no edges to nodes in $(V_0 \setminus OPT)$, otherwise these edges are not covered by OPT. IF $w(B_I) \geq w(L_O)$ then we can replace B_I by L_O in OPT and get another OPTIMAL solution satisfying (a). If $w(B_I) < w(L_O)$ then we can pick an ϵ and then modify the fractional solution by reducing the X_v values of the nodes in L_O by ϵ and increasing the fractional X_v values of the nodes in B_I to produce a lower weight fractional solution, contradicting the assumption that we started off with an optimal LP solution. □

Using this theorem, lets us concentrate our attention on finding a vertex cover in the subgraph induced by the vertices in $V_{\frac{1}{2}}$. If we find a cover S in this subgraph, then $S \cup V_1$ gives us a cover in the original graph. Moreover, if we can prove bounds on the weight of S then we could use this to derive an approximation factor better than 2. Note that the optimal solution in $V_{\frac{1}{2}}$ has weight $\geq \frac{1}{2} w(V_{\frac{1}{2}})$.