Sufficient Condition for a 4-Dimensional Vector Orbi-Space to Admit a Faithful Symplectic SU(2) Action

Sorelle Friedler
Janet Talvacchia
Summer 2003

Abstract
In this paper we state a sufficient condition for the existence of a 4-dimensional vector orbi-space which admits a faithful, symplectic SU(2) action.

1 Introduction
Classifying all Hamiltonian SU(2) actions on manifolds is a hard unsolved problem. We begin by looking at SU(2) actions on vector orbi-spaces since they contain all the local information. In this paper we look at faithful symplectic SU(2) actions on 4-dimensional vector orbi-spaces and have found the following sufficient condition:

Main Theorem 1. If \( \Gamma \) is a finite subgroup of the center of SU(2), there is a 4-dimensional vector orbi-space \( V/\Gamma \) which admits a symplectic, faithful SU(2) action.

2 Background
2.1 Vector Space
A vector space \( V \) over a field \( F \) is a set together with two laws of composition:
1. \( V \times V \to V, \; v, w \mapsto v + w \) (addition)
2. \( F \times V \to V, \; c, v \mapsto cv \) (scaler multiplication)
and satisfying the following axioms:
1. addition makes \( V \) into a commutative group \( V^+ \).
2. scaler multiplication is associative with multiplication in \( F \):
   \( (ab)v = a(bv) \; \forall \; a, b \in F \) and \( v \in V \)
3. the element 1 acts as the identity: \( 1v = v \; \forall \; v \in V \)
4. two distributive laws hold:
   \( (a + b)v = av + bw \) and \( a(v + w) = av + aw \; \forall \; a, b \in F \) and \( v, w \in V \)
2.2 Bilinear Form

A bilinear form is a form on a vector space $V$ that is a function of 2 variables on $V$ with values in the field $F$, $V \times V \rightarrow F$. $f$ satisfies the bilinear axioms:

1. $f(v_1 + v_2, w) = f(v_1, w) + f(v_2, w)$
2. $f(v, w_1 + w_2) = f(v, w_1) + f(v, w_2)$
3. $f(cv, w) = cf(v, w)$
4. $f(v, cw) = cf(v, w)$

notation: $< v, w >$

2.3 Skew-symmetric Bilinear Form

Intuitively, a skew-symmetric bilinear form is one such that

$< v, w >= - < w, v >$

(since a symmetric bilinear form has $< v, w > = < w, v >$).

However, this definition, while it is useful, does not hold for characteristic 2. The universal definition is that a bilinear form is skew-symmetric if

$< v, w >= 0 \forall v \in V$

2.4 Nondegenerate Bilinear Form

A nondegenerate bilinear form is any bilinear form such that

$\forall v \in V < v, w >= 0 \forall w \in V$ implies that $v = 0$

2.5 Matrix Representation of a Bilinear Form

Take a basis for $V$ with $<,>$ a bilinear form on $V$. Let $B = (b_1, b_2, \ldots b_n)$ be the basis. The matrix of the form with respect to the basis is $A = (a_{ij})$ where $a_{ij} = < b_i, b_j >$.

The standard skew-symmetric form represented as a matrix is: $J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$

2.6 The Symplectic Group

The symplectic group is the stabilizer of $J$ as given above.

$SP_{2n}(\mathbb{R}) = \{ P \in GL_{2n}(\mathbb{R}) | P^t JP = J \}$

The complex symplectic group is defined similarly. Note that all symplectic matrices have determinant 1.

2.7 Symplectic Vector Space

A symplectic vector space is a pair $(v, \omega)$ where $V$ is a finite dimensional real vector space and $\omega$ is a nondegenerate skew-symmetric bilinear form $\omega : V \times V \rightarrow \mathbb{R}$. Since the bilinear form is nondegenerate, the dimension of the symplectic vector space is always even. All symplectic vector spaces with the same dimension are isomorphic.

example: $(\mathbb{R}^{2n}, \omega)$ where $\omega$ has the matrix representation $J.$
2.8 Quotient Space
Let \( V \) be a vector space over a field \( F \) and \( W \) be a subspace of \( V \). Then \( V/W \) is a vector space over \( F \) and the quotient space of \( V \) by \( W \) with
1. \((v_1 + W) + (v_2 + W) = (v_1 + v_2) + W\)
2. \(\alpha(v_1 + W) = \alpha v_1 + W\)
given \( v_1 + W, v_2 + W \in V/W \) and \( \alpha \in F \)

2.9 Symplectic Vector Orbi-Space
The quotient space \( V/\Gamma \) where \( V \) is a symplectic vector space and \( \Gamma \) is a finite subgroup of the symplectic group \( \text{SP}(V) \).

2.10 Unitary Matrix
\( P \) is a unitary matrix if \( P^*P = I \) where \( I \) is the identity matrix (or \( P^* = P^{-1} \)) and \( P^* \) is the matrix adjoint, \( P^* = \overline{P}^T \).

2.11 Unitary Group \( U_n \)
\( U_n = \{P|P^*P = I\} \)
(This is the group of matrices representing changes of basis which leave the hermitian dot product \( X^*Y \) invariant. [1](p27))

2.12 Special Matrices
Special matrix groups are subgroups of matrix groups that have determinant 1.

2.13 Special Linear Group
The special linear group (\( SL_n(\mathbb{R}) \)) is the group of \( n \times n \) matrices with determinant 1 and entries in \( \mathbb{R} \). (A complex group can be defined analogously.)

2.14 Special Unitary Group \( SU(n) \)
\( SL_n(\mathbb{C}) \cap U_n \)

2.15 Group Action on a Set
A group \( G \) is said to act (or operate) on the set \( S \) if there exists a map \((g, x) \mapsto gx \) of \( G \times S \) into \( S \) satisfying:
1. \( 1x = x, x \in S \)
2. \((g_1, g_2)x = g_1(g_2x)\)
([6], p72)
3 Preliminary Theorems

**Theorem 1.** [9] Let \( \rho : H \to SP(V/\Gamma) \) be a faithful symplectic representation of a compact Lie group \( H \) on a symplectic vector orbi-space \( V/\Gamma \), and let \( N(\Gamma) \) denote the normalizer of \( \Gamma \) in \( SP(V) \). The representation \( \rho \) and the short exact sequence \( 1 \to \Gamma \to N(\Gamma) \to SP(V/\Gamma) \to 1 \) give rise to the pull-back extension \( \pi : \hat{H} \to H \) and the faithful (symplectic) pull-back representation \( \hat{\rho} : \hat{H} \to N(\Gamma) \subset SP(V) \) so that \( \Gamma \) is naturally a subgroup of \( \hat{H} \), and the following diagram is exact and commutes.

\[
\begin{array}{cccccc}
1 & \to & \Gamma & \to & \hat{H} & \xrightarrow{\pi} & H & \to & 1 \\
\| & & \downarrow{\hat{\rho}} & & \downarrow{\rho} & & & & \\
1 & \to & \Gamma & \to & N(\Gamma) & \xrightarrow{\mu} & SP(V/\Gamma) & \to & 1
\end{array}
\]

Conversely, given a Lie group \( \hat{H} \in SP(V) \), a symplectic representation \( \hat{\rho} : \hat{H} \to SP(V) \) of \( \hat{H} \) on a symplectic vector space \( V \) and a finite normal subgroup \( \Gamma \) of \( \hat{H} \) such that \( \hat{\rho}(\Gamma) = \Gamma \), there exists a symplectic orbi-representation \( \rho : H \to SP(V/\Gamma) \) of the quotient \( H = \hat{H}/\Gamma \) making the above diagram commute.

**Proof.** Let \( \rho : H \to N(\Gamma)/\Gamma \) be a faithful symplectic representation and let \( \mu : N(\Gamma) \to N(\Gamma)/\Gamma \) be defined \( a \mapsto a\Gamma \). Let \( \hat{H} = \mu^{-1}(\rho(H)) \).

The group \( \hat{H} \) is a subgroup of \( N(\Gamma) \) as \( h, k \in \hat{H} \Rightarrow [h] \in SP(V/\Gamma), [k] \in SP(V/\Gamma), \rho^{-1}([h]) \in H \) and \( \rho^{-1}([k]) \in H \). Since \( H \) is a group,

\[
\rho^{-1}([h])\rho^{-1}([k])^{-1} \in H \Rightarrow [h][k]^{-1} \in SP(V/\Gamma) \\
\Rightarrow [h][k]^{-1} \in SP(V/\Gamma) \\
\Rightarrow hk^{-1} \in \hat{H}
\]

Thus \( \hat{H} \) is a subgroup.

Furthermore \( \hat{H} \) is a Lie group. Since multiplication in the Lie group \( SP(V) \) is smooth, the function \( \pi : \hat{h} \to H \) defined \( a \mapsto a\Gamma \) is continuous. Therefore if we consider \( H \) to be in \( SP(V) \), since \( H \) is closed in \( SP(V) \) being that \( H \) is compact, \( \pi^{-1}(H) = \hat{H} \) is also closed. By SOME THEOREM \( \hat{H} \) is a Lie group.

Thus, let \( \pi : \hat{H} \to H \) be defined \( a \mapsto \rho^{-1}(\mu(a)) \). Then \( \hat{\rho} \) is an inclusion, it is seen that \( \rho \circ \pi = \mu \circ \hat{\rho} \), \( \Gamma \) is a subgroup of \( \hat{H} \) and the sequence is exact.

Conversely, given a group \( \hat{H} \subset SP(V) \), a symplectic representation \( \hat{\rho} : \hat{H} \to SP(V) \) of \( \hat{H} \) on a symplectic vector space \( V \) and a finite normal subgroup \( \Gamma \subset SP(V) \) of \( \hat{H} \) such that \( \hat{\rho}(\Gamma) = \Gamma \), we have \( \hat{\rho}(\hat{H}) \subset N(\Gamma) \subset SP(V) \). This follows since for all \( h \in \hat{H} \), \( h\Gamma h^{-1} = \Gamma \) and since \( \rho \) is a homomorphism \( \rho(h\Gamma h^{-1}) = \rho(\Gamma) = \Gamma \Rightarrow \rho(h)\rho(\Gamma)\rho(h)^{-1} = \Gamma \Rightarrow \rho(h) = N(\Gamma) \).

If we let \( \pi : \hat{H} \to H \) be defined \( h \mapsto h\Gamma \) and \( \mu : N(\Gamma) \to SP(V/\Gamma) \) be defined \( h \mapsto h\Gamma \) then we can let \( \rho : H \to SP(V/\Gamma) \) be defined \( [h] \mapsto \mu(\hat{\rho}(h)) \). It is obvious that \( \rho \circ \pi = \mu \circ \hat{\rho} \) and that the diagram commutes and is exact.

**Theorem 2.** If \( \Gamma \subset Z(G) \) (the center of \( G \)) then \( \Gamma \) is a normal subgroup of \( G \).

**Proof.** A subgroup \( N \) of a group \( G \) is called a normal subgroup if it has the property that for all \( a \in N \) and \( b \in G \), \( bab^{-1} \in N \). So we want to show that for all \( a \in \Gamma \) and \( b \in G \), \( bab^{-1} \in \Gamma \):

Pick some \( b \in G \) and \( a \in \Gamma \). Is \( bab^{-1} \in \Gamma \)? Since \( a \) is in the center of \( G \), \( a \) commutes with \( b \) and \( b^{-1} \) since both are in \( G \).

So \( bab^{-1} = bb^{-1}a = a \) and \( a \in \Gamma \).

**Theorem 3.** If \( \hat{G} = \langle G, \Gamma \rangle = \{tv : t \in G, v \in \Gamma \} \) where elements of \( G \) and \( \Gamma \) are represented by square matrices of the same dimension and \( \Gamma \) is normal in \( G \), \( \Gamma \) is normal in \( \hat{G} \).
Proof. Let \( a \in \Gamma \) and \( tv \in \hat{G} \). If \( tv(tv)^{-1} \in \hat{G} \), \( \Gamma \) is normal in \( \hat{G} \).

\[
(tva(tv))^{-1} = tva^{-1}t^{-1}
\]

since \( v \) and \( t \) are matrices

\[
\varphi v^{-1} \in \Gamma \text{ since } v, a \in \Gamma
\]

let \( vav^{-1} = b \in \Gamma \)

\[
= tbt^{-1}
\]

\[
= ttt^{-1}b = b
\]

So \( \Gamma \) is normal in \( \hat{G} \).

\[\square\]

**Theorem 4.**

\[\varphi : \begin{pmatrix} a & b & a' & b' \\ -b & a & b' & -a' \\ -a' & -b' & a & b \\ -b' & a' & -b & a \end{pmatrix}\]

with \( a^2 + b^2 + a'^2 + b'^2 = 1 \in SP(4, \mathbb{R}) \rightarrow \begin{pmatrix} a + d'i & b + y'i \\ -b + y'i & a - d'i \end{pmatrix} \in SU(2) \)

is an isomorphism.

**Proof.** Let

\[A = \begin{pmatrix} a & b & a' & b' \\ -b & a & b' & -a' \\ -a' & -b' & a & b \\ -b' & a' & -b & a \end{pmatrix}\]

and \( C = \begin{pmatrix} c & d & c' & d' \\ -d & c & d' & -c \\ -c' & -d' & c & d \\ -d' & c' & -d & c \end{pmatrix}\)

with \( a^2 + b^2 + a'^2 + b'^2 = 1 \) and \( c^2 + d^2 + c'^2 + d'^2 = 1 \)

First, \( A \in SP(4, \mathbb{R}) \):

\[A^t J A = \begin{pmatrix} a & b & a' & b' \\ -b & a & b' & -a' \\ -a' & -b' & a & b \\ -b' & a' & -b & a \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & a' & b' \\ -b & a & b' & -a' \\ -a' & -b' & a & b \\ -b' & a' & -b & a \end{pmatrix} = \]

\[
\begin{pmatrix} a' & b' & a & b \\ b' & a & b & a' \\ -a & b & a' & b \\ b & a & b & a' \end{pmatrix}
\]

since we have the condition \( a^2 + b^2 + a'^2 + b'^2 = 1 \) we get

\[
\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = J
\]

So \( A \in SP(4, \mathbb{R}) \).

\( \varphi \) is a homomorphism:

We want to show that \( \varphi(AC) = \varphi(A)\varphi(C) \).

\[
\varphi(AC) = \varphi \left( \begin{pmatrix} a & b & a' & b' \\ -b & a & b' & -a' \\ -a' & -b' & a & b \\ -b' & a' & -b & a \end{pmatrix} \begin{pmatrix} c & d & c' & d' \\ -d & c & d' & -c \\ -c' & -d' & c & d \\ -d' & c' & -d & c \end{pmatrix} \right) = \]

\[\text{[Expression]}\]

5
\[
\varphi \left( \begin{pmatrix}
ac - a'c' - bd' - bd' \\
bd + ac - ad' + be' \\
ac' + bd' + a'c - d'd \\
ad' - bc' + a'd + b'c
\end{pmatrix}
\right) = \begin{pmatrix}
ac - a'c' - bd' - bd' + (ad + bc - a'd' + b'e')i \\
bd + ac - ad' - bc' + bd' + (ad - bc - a'd' + b'c)i \\
ac' + bd' + a'c - d'd + (ad' - bc' + a'd + b'c)i \\
ad' - bc' + a'd' - bc' + bd' + ac
\end{pmatrix}
\]

\[
(ace - bde - b'd = bd'd') + (ace' + bd'e + b'd')i = (ad + bc - a'd' + b'e')i + (ad' - bc' + a'd + b'c)i
\]

\[
\varphi = \begin{pmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{pmatrix} \in SU(2)
\]

\[
\varphi(\alpha, \beta) = \varphi(\alpha, \beta) = \varphi(\alpha, \beta)
\]

\[
\varphi \text{ is obviously injective.}
\]

Is \( \varphi \) surjective?

Pick some element \( \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in SU(2) \). We want to show that there is some element in \( SP(4, \mathbb{R}) \) which maps to \( \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \). Pick \( \begin{pmatrix} a & b & a' & b' \\ -b & a & b' & -a' \\ -a' & -b' & a & b \\ -b' & a' & -b & a \end{pmatrix} \) such that \( a + a'i = \alpha \) and \( b + b'i = \beta \) then since

\[
\det \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = a^2 + b^2 + a'^2 + b'^2
\]

So \( \varphi \) is surjective.

\( \Box \)

4 Main Theorem

Main Theorem 1. If \( \Gamma \) is a finite subgroup of the center of \( SU(2) \), there is a 4-dimensional vector orbi-space \( V/\Gamma \) which admits a symplectic, faithful \( SU(2) \) action.

Proof. Let \( \tilde{\Gamma} \) be a finite subgroup of the center of \( SU(2) \) and \( \varphi \) as described in Theorem 4. Let \( \Gamma = \varphi^{-1}(\tilde{\Gamma}) \in SP(4, \mathbb{R}) \).

Let \( SU(2) = \left\{ \begin{pmatrix} a & b & a' & b' \\ -b & a & b' & -a' \\ -a' & -b' & a & b \\ -b' & a' & -b & a \end{pmatrix} \in SP(4, \mathbb{R}) \right\} \).

Let \( \overline{SU}(2) = \left\{ tv : t \in SU(2), v \in \Gamma \right\} \). \( \overline{SU}(2) \in SP(4, \mathbb{R}) \) since \( \Gamma \subset SP(4, \mathbb{R}) \) and \( SU(2) \subset \overline{SU}(2) \).
$SP(4, \mathbb{R})$. $\Gamma$ is normal in $SU(2)$ by Theorems 2 and 3 since $\Gamma \subset Z(SU(2))$.

Thus by Theorem 1, since we have a symplectic representation of $\widehat{SU}(2)$ on the vector orbi-space $\mathbb{R}^4/\Gamma$, we have the following diagram which is exact and commutes:

$$
\begin{array}{cccc}
1 & \rightarrow & \Gamma & \rightarrow & \widehat{SU}(2) & \xrightarrow{\pi} & SU(2) & \rightarrow & 1 \\
\| & \downarrow & \hat{\rho} & & \downarrow & \rho \\
1 & \rightarrow & \Gamma & \rightarrow & N(\Gamma) & \xrightarrow{\iota} & SP(V/\Gamma) & \rightarrow & 1
\end{array}
$$

Thus there is a symplectic representation of $SU(2)$ on the vector orbi-space $\mathbb{R}^4/\Gamma$. Thus the vector orbi-space admits an $SU(2)$ action.

References


