NEXT YEAR: PROVE that tracing/mining out a variable from a convex function preserves convexity.

This homework should be scanned/imaged and uploaded to Elms, or optionally turned into me in class. To get full credit, you must print this homework and write your solutions in the space provided. Use either my slides, or Boyd §3.2 as a reference.

1. (20 points) (a) Give an example of a convex function that is not proper.

\[ \text{Solution: } f(x) = \infty \]

(b) Give an example of a convex function that is bounded below, but has no minimizer.

\[ \text{Solution: } f(x) = e^x \]

(c) Give an example of a convex function that is not closed. Hint: Suppose I gave you a convex function that holds beer. What could you do to make all the beer leak out?

\[ \text{Solution: } \text{Let } f : \left[ -\infty, 0 \right] \rightarrow \mathbb{R} \text{ be defined by} \]
\[ f(x) = \begin{cases} 
0, & \text{if } x < 0 \\
1, & \text{if } x = 0 
\end{cases} \]

(d) Give an example of a convex function that has unbounded level sets, but has at least one minimizer.

\[ \text{Solution: } f(x, y) = x^2 \]
2. (50 points) For this problem, you may use any of the results/rules from the lecture slides.

(a) Consider the set of probability functions \( \{ f | f \geq 0, \text{and } \int f = 1 \} \). Is this a convex set?

Solution: Yes. For \( f, g \in C \),
\[
\int \theta f + (1 - \theta)g = \theta \int f + (1 - \theta) \int g = \theta + (1 - \theta) = 1
\]

(b) Consider the function \( M_5 : \mathbb{R}^{10} \rightarrow \mathbb{R} \), where \( M_5(z) \) returns the sum of the 5 largest entries in \( x \). Is this function convex? Why?

Solution: Given a subset \( S \) of 5 indices, let \( M_S(x) \) denote the sum of the corresponding entries is \( x \). Then
\[
M_5 = \max_S M_S(x)
\]
where the max is over all subsets of size 5.

(c) Consider the set of low-rank matrices \( \{ A | \text{rank}(A) < k \} \). Is this set convex? Why?

Solution: No. Here’s a counter-example for \( k = 1 \):
\[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
(d) Let \( C \in \mathbb{R}^n \) be a compact (closed and bounded) convex set. For a scalar \( x \), let \( C(x) \) be the set of all points in \( C \) whose first coordinate is closer to \( x \) than any other point. More precisely: \( C(x) = \{ a \in C \mid \forall b \in C, |a_1 - x| \leq |b_1 - x| \} \). Is \( C(x) \) a convex set for any choice of \( C \) and \( x \)? Why?

**Solution:** Choose some \( C \) and \( x \). Define \( f(a) = |a_1 - x| \). This is a convex function. The set \( C(x) \) is the set of minimizers of \( f \), and is therefore convex.

(e) Is the set \( C = \{ (x,y) \mid \|x\| \leq y \} \) convex? Is it a cone? Why? What about the set \( C = \{ (x,y) \mid \|x\|^2 \leq y \} \)? Is this convex? Is this a cone?

**Solution:** Let \( (x,y) \) and \( (x',y') \) be in \( C \). The first set is convex because

\[
\|\theta x' + (1-\theta)x\| \leq \theta \|x'\| + (1-\theta)\|x\| \leq \theta y' + (1-\theta)y.
\]

It is a cone because

\[
\|x\| \leq y \rightarrow \|\alpha x\| = \alpha \|x\| \leq \alpha y
\]

for non-negative \( y \).

The second set is convex by a similar argument (using Jensen's inequality on \( \|\cdot\|^2 \)). However, this particular set is not always a cone. Look in 1 dimension and assume the 2-norm. Then \( (1,1) \) is in the set, but \( (2,2) \) is not.
3. (10 points) Suppose $f$ is strongly convex. Show that the gradient of $f$ is strongly monotone, i.e., there is a constant $m > 0$ with

$$\langle x - y, \nabla f(x) - \nabla f(y) \rangle \geq m\|x - y\|^2.$$ 

**Solution:** By the definition of strong convexity

$$f(y) \geq f(x) + (y - x)^T \nabla f(x) + \frac{m}{2} \|y - x\|^2.$$ 

We also have

$$f(x) \geq f(y) + (x - y)^T \nabla f(y) + \frac{m}{2} \|x - y\|^2.$$ 

Adding these together and cancelling terms yields

$$0 \geq (x - y)^T (\nabla f(y) - \nabla f(x)) + m\|x - y\|^2,$$

which re-arranges to yield the result.

4. (20 points) Suppose that $g(x)$ is convex and $h(x)$ is concave (i.e. $-h(x)$ is convex). Suppose we restrict both functions into a closed, convex set $C$ such that both $g(x)$ and $h(x)$ are always positive when $x \in C$. Prove that the function $f(x) = g(x)/h(x)$ is quasi-convex. (note: It follows that every local minimum is also global)

**Solution:** Choose some $\alpha \geq 0$ and consider the sub-level set $S_\alpha = \{x | f(x) \leq \alpha\}$. If $x \in S_\alpha$, then

$$g(x)/h(x) \leq \alpha \quad \rightarrow \quad g(x) \leq \alpha h(x) \quad \rightarrow \quad g(x) - \alpha h(x) \leq 0.$$ 

But $-h(x)$ is convex and $\alpha \geq 0$, and so $g(x) - \alpha h(x)$ is convex. Because the set $S_\alpha$ is a sub-level set of the convex function $g(x) - \alpha h(x)$, it is convex.