1. Consider the “monotropic” program

\[
\begin{aligned}
\text{minimize} & \quad \|x\|_{\infty} \\
\text{subject to} & \quad Ax = b.
\end{aligned}
\]  

\(\text{(a)}\) Write this as an unconstrained (or implicitly constrained) problem using the characteristic function of the zero vector \(\chi_0(z)\). This function is zero if it’s argument is zero, and infinite otherwise.

**Solution:** Put your solution here

\(\text{(b)}\) What is the conjugate of \(f(z) = \|z\|_{\infty}\)?

**Solution:** Put your solution here

\(\text{(c)}\) What is the conjugate of \(g(z) = \chi_0(z)\)?

**Solution:** Put your solution here

\(\text{(d)}\) Using the conjugate functions, write down the dual of (1).

**Solution:** Put your solution here

2. Consider the linear program

\[
\begin{aligned}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0.
\end{aligned}
\]  

\(\text{(a)}\) Write the optimality conditions for this problem (i.e., the KKT system).

**Solution:** Put your solution here

\(\text{(b)}\) Write the Lagrangian for this problem.

**Solution:** Put your solution here

\(\text{(c)}\) Minimize out the primal variables in the Lagrangian, and write the dual formulation of this linear program.
3. Consider the problem

$$\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g(x) \leq 0.
\end{align*}$$

Let $x_0$ be a solution to this problem, and $\lambda_0$ be the corresponding optimal Lagrange multiplier. Now, define a perturbed problem

$$\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g(x) \leq \epsilon
\end{align*}$$

where $\epsilon$ is a vector. Let $x_\epsilon$ be a solution to the perturbed problem. Note, if we put large negative values in $\epsilon$, then the constraint set gets smaller, and we expect the corresponding value of $f(x_\epsilon)$ to increase.

Prove the “sensitivity bound”

$$f(x_0) - \lambda_0^T \epsilon \leq f(x_\epsilon).$$

This bound shows that the Lagrange multipliers determine how much the objective increases as the vector $\epsilon$ becomes more negative.

4. (a) Let $\text{prox}_f(x,t) = \arg\min_z f(z) + \frac{1}{2t} \|z - x\|^2$. Prove the “Moreau decomposition” identity

$$x = \text{prox}_f(x,t) + t \text{prox}_{f^*}(x/t, 1/t).$$

(b) Using your result from part (a), prove that

$$\text{prox}_{|x|}(x,t) = x - \max\{\min\{x,t\}, -t\}$$

where $|x|$ denotes the 1-norm of $x$, and “min” and “max” are applied element-wise. You may use the identity $[af]^*(x) = af^*(x/a)$ where $f^*$ denotes the conjugate of $f$, and $a$ is a scalar.