Your name: (put name here)

Answer the following questions. Latex your solutions into this document, and submit a pdf on Elms.

1. Consider the "monotropic" program

$$\begin{aligned} \text{minimize} \quad & \|x\|_\infty \\ \text{subject to} \quad & Ax = b. \end{aligned} \tag{1}$$

   (a) Write this as an unconstrained (or implicitly constrained) problem using the characteristic function of the zero vector $\chi_0(z)$. This function is zero if it's argument is zero, and infinite otherwise.

   Solution: Put your solution here

   (b) What is the conjugate of $f(z) = \|z\|_\infty$?

   Solution: Put your solution here

   (c) What is the conjugate of $g(z) = \chi_0(z)$?

   Solution: Put your solution here

   (d) Using the conjugate functions, write down the dual of (1).

   Solution: Put your solution here

2. Consider the linear program

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

   (a) Write the optimality conditions for this problem (i.e., the KKT system).

   Solution: Put your solution here

   (b) Write the Lagrangian for this problem.

   Solution: Put your solution here

   (c) Minimize out the primal variables in the Lagrangian, and write the dual formulation of this linear program.

Solution: Put your solution here

3. Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g(x) \le 0. \end{aligned}$$

Let x_0 be a solution to this problem, and λ_0 be the corresponding optimal Lagrange multiplier. Now, define a perturbed problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g(x) \le \epsilon \end{aligned}$$

where ϵ is a vector. Let x_ϵ be a solution to the perturbed problem. Note, if we put large negative values in ϵ, then the constraint set gets smaller, and we expect the corresponding value of $f(x_\epsilon)$ to increase.

Prove the "sensitivity bound"

$$f(x_0) - \lambda_0^T \epsilon \le f(x_\epsilon).$$

This bound shows that the Lagrange multipliers determine how much the objective increases as the vector ϵ becomes more negative.

Solution: Put your solution here

4. (a) Let $\text{prox}_f(x,t) = \text{argmin}_z\, f(z) + \frac{1}{2t}\|z - x\|^2$. Prove the "Moreau decomposition" identity

$$x = \text{prox}_f(x,t) + t\, \text{prox}_{f^*}(x/t, 1/t).$$

Solution: Put your solution here

(b) Using your result from part (a), prove that

$$\text{prox}_{|x|}(x,t) = x - \max\{\min\{x,t\}, -t\}$$

where $|x|$ denotes the 1-norm of x, and "min" and "max" are applied element-wise. You may use the identity $[af]^*(x) = af^*(x/a)$ where f^* denotes the conjugate of f, and a is a scalar.

Solution: Put your solution here