LINEAR PROGRAMMING

THE ORIGINAL "BIG DATA"

1948: Soviets blockade West Berlin

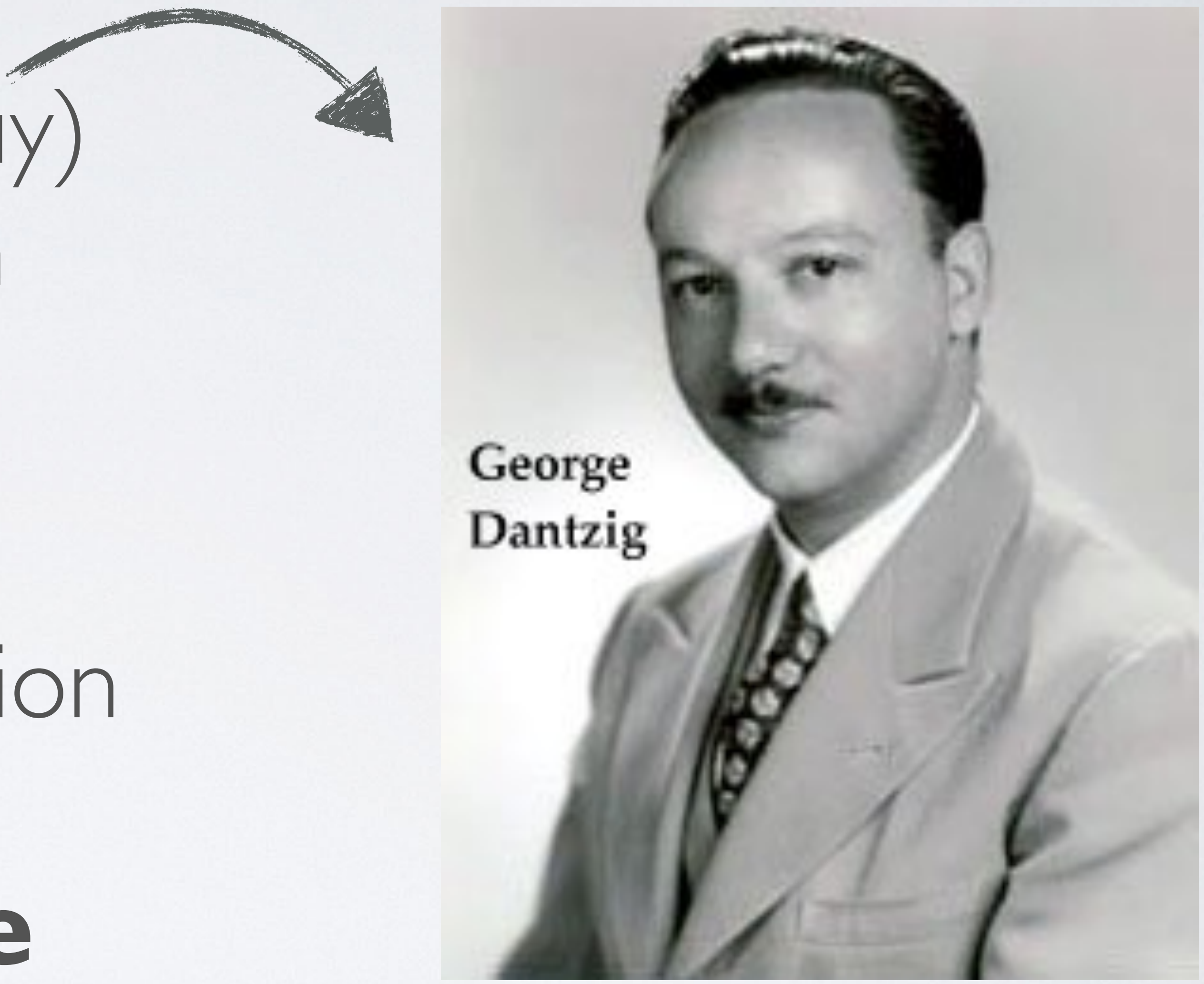
- One week later: airlift begins
- Big operation: $250 million
- 3 Billion today in 2 weeks

Difficult scheduling problem!

- 3 routes
- 5 nations
- Plane landed every 30 seconds
- 2,346,000 tons of freight

THE ORIGINAL "BIG DATA"

- George B. Dantzig (this guy) appointed to solve problem
- LP formulation
- Simplex method



- US Air Force implementation
- IBM calculation machines
- Conclusion: **Intractable**

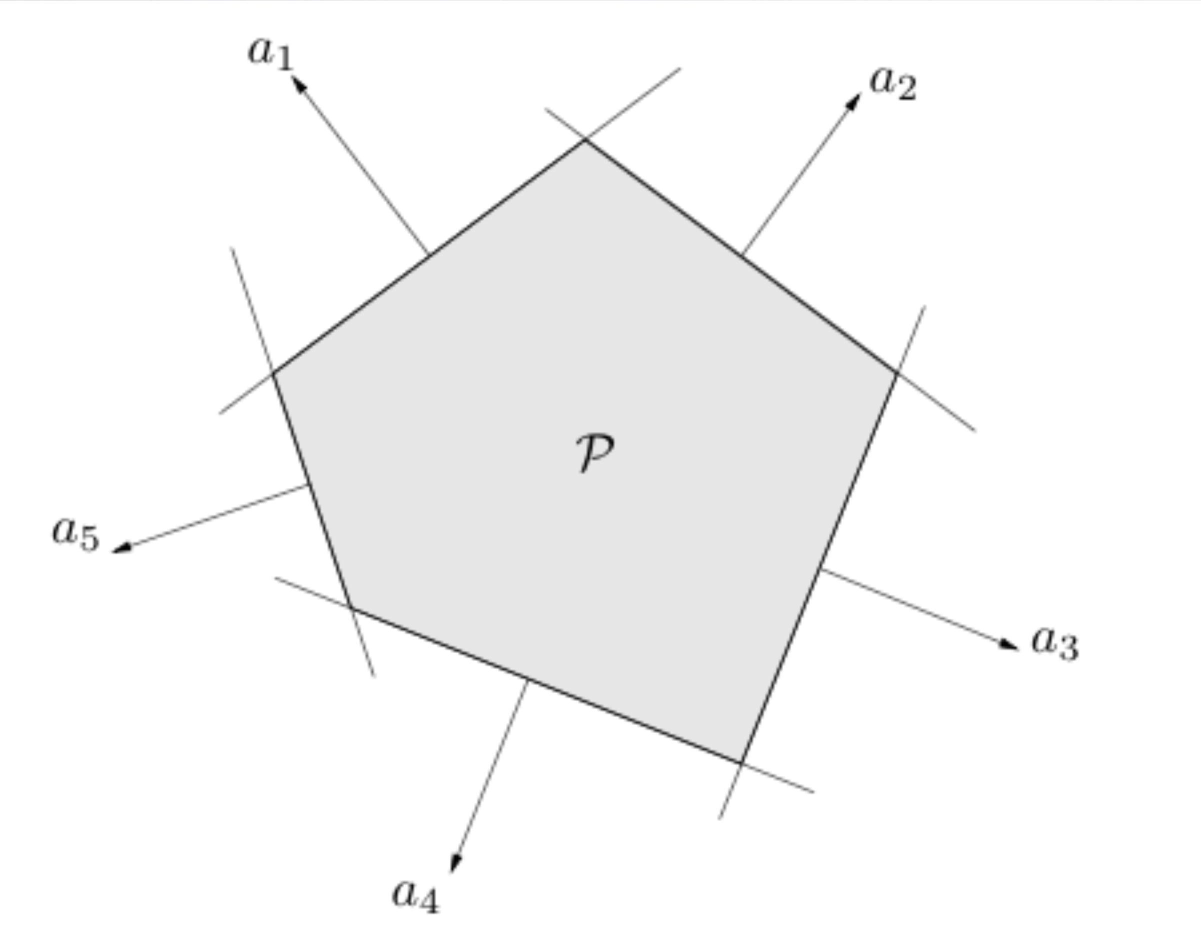
50 unknowns

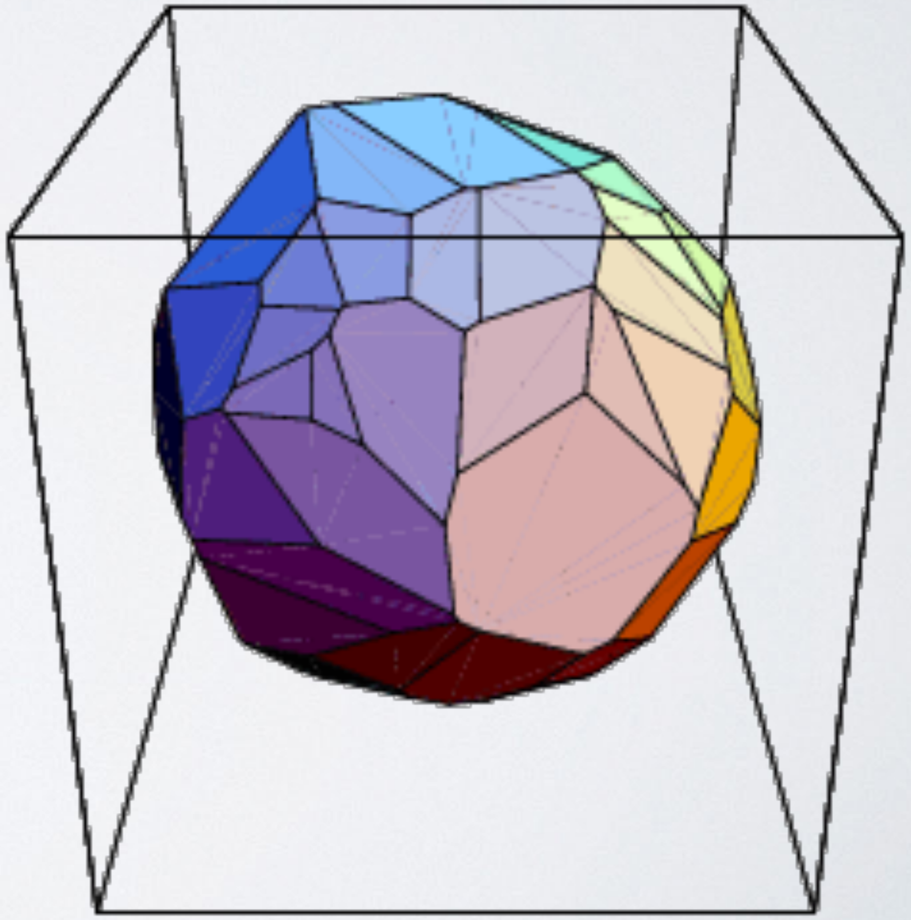
Source: Klein, 2013

SIMPLEX

$$\{x : \ Ax \leq b\}$$

Is it convex? Why?



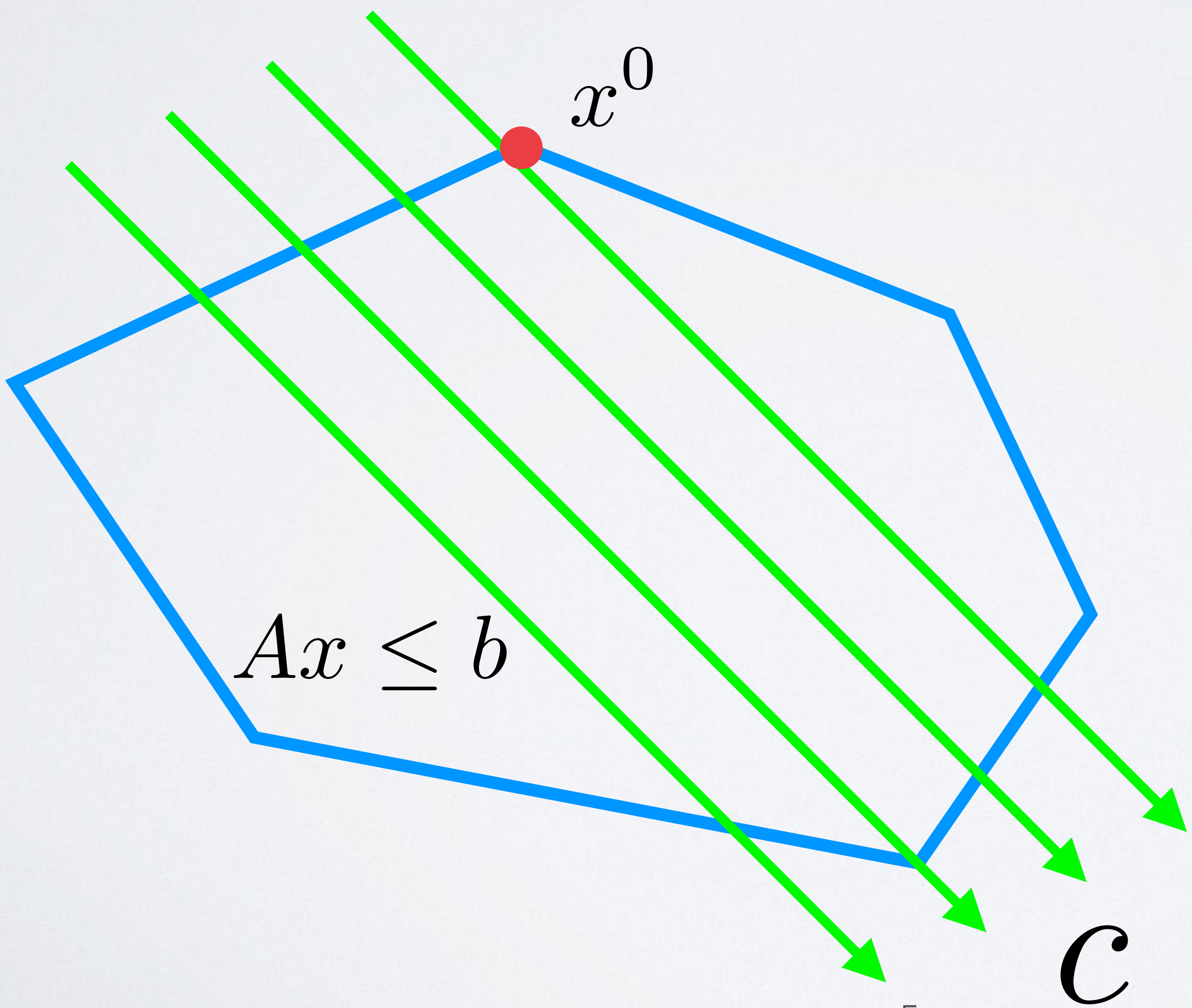


SIMPLEX METHOD

$$\text{maximize} \quad c^T x$$
$$\text{subject to} \quad Ax \leq b$$

Solution is always on vertex, why?

How could you solve this?



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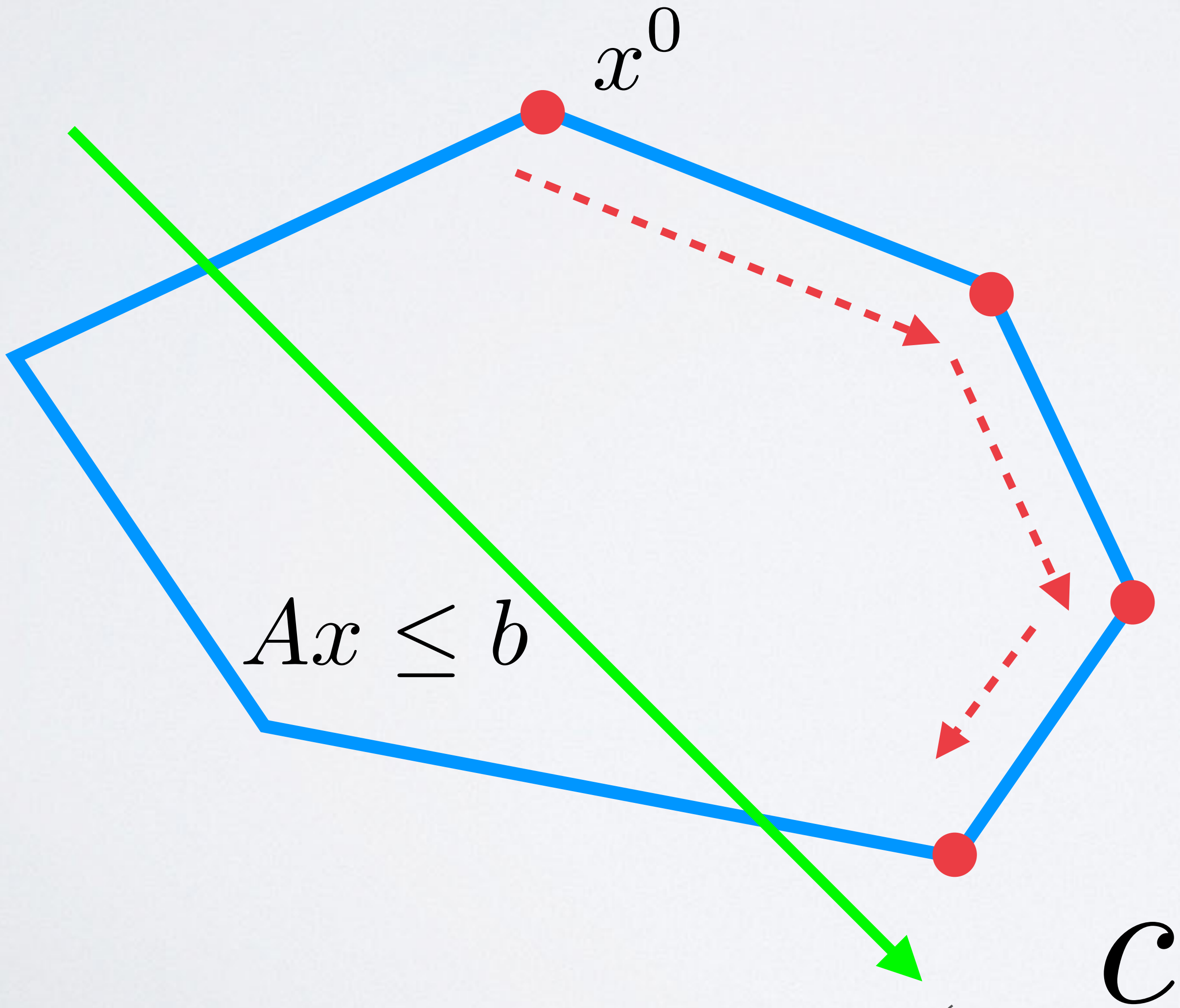
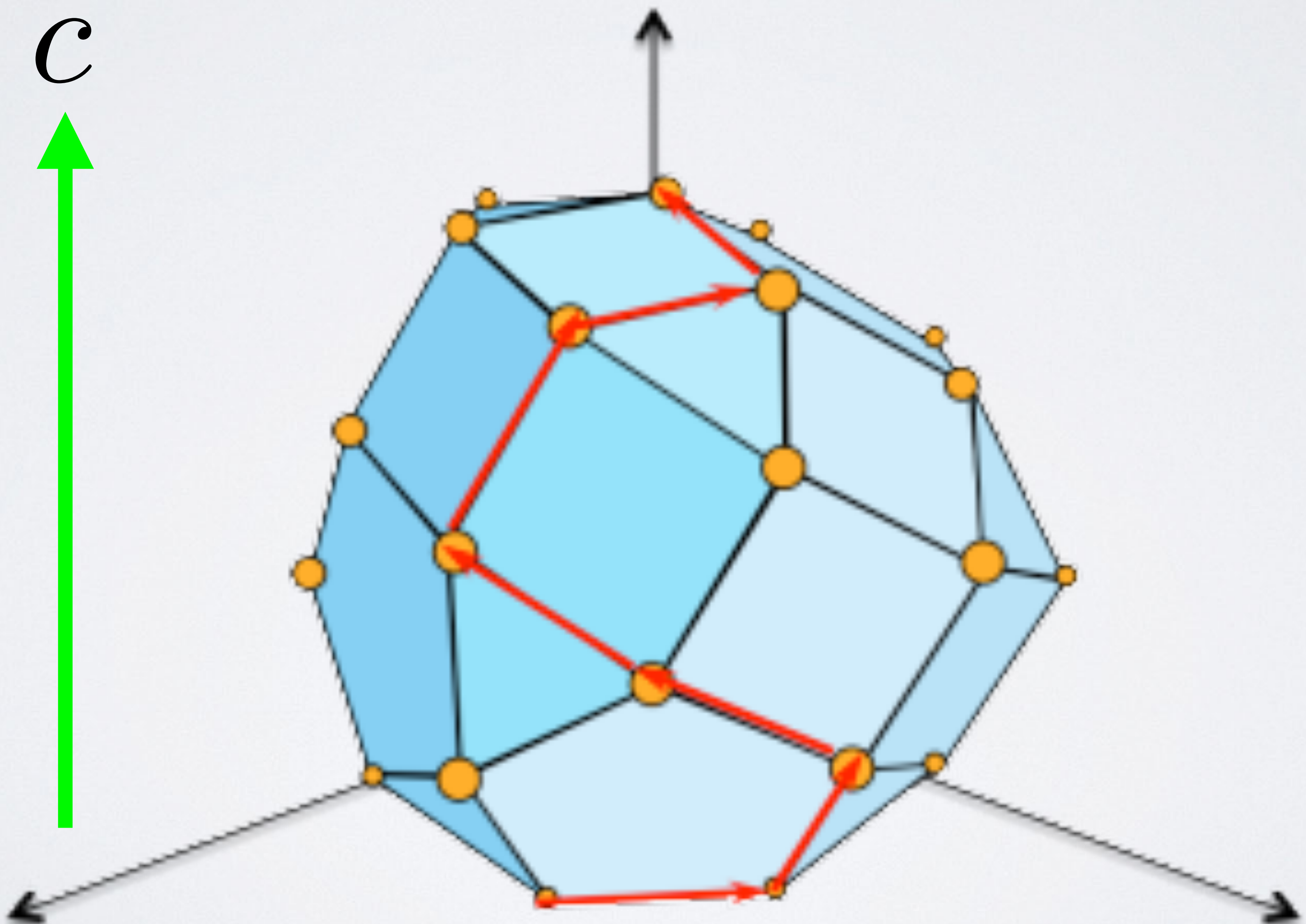


How could you solve this?

Gradient descent!

SIMPLEX METHOD

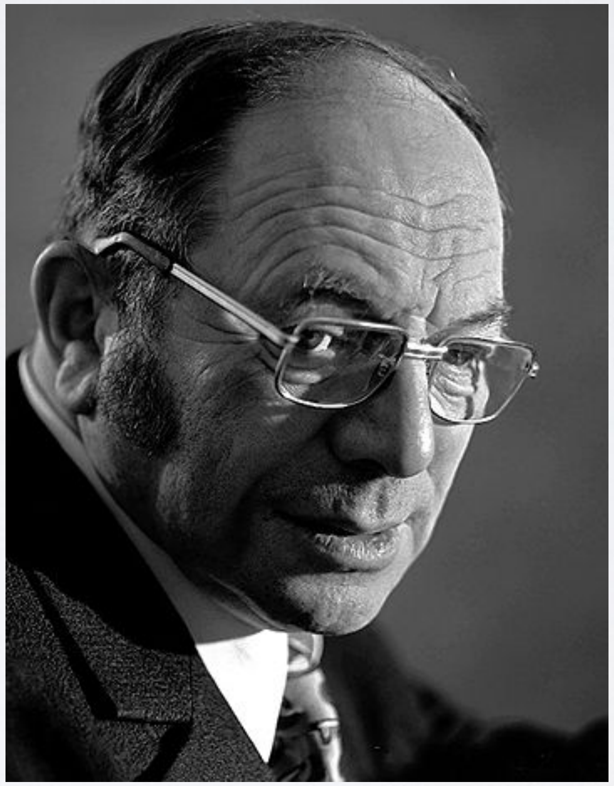
$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

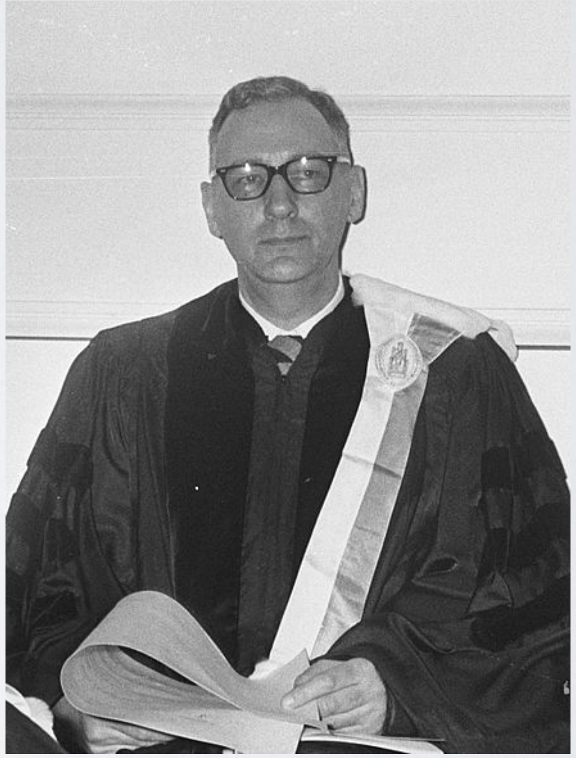


YOU JUST WON A NOBEL PRIZE!

(if you lived 60 years ago)

Dantzig wins
National Medal of Science



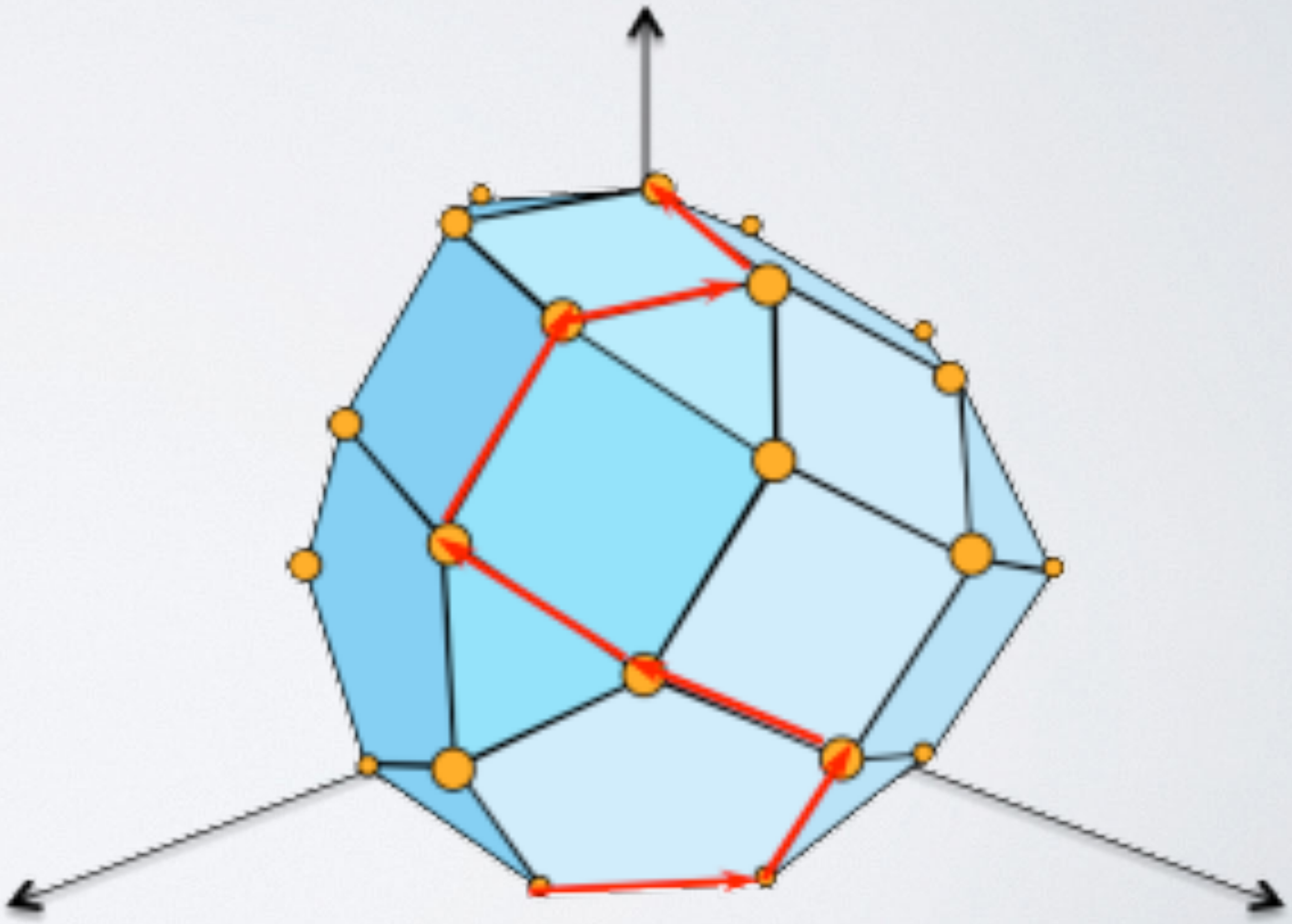


Leonid Kantorovich, Tjalling Koopmans
win Nobel Prize

THE BIG IDEA

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Q: How many constraints are active at each vertex?



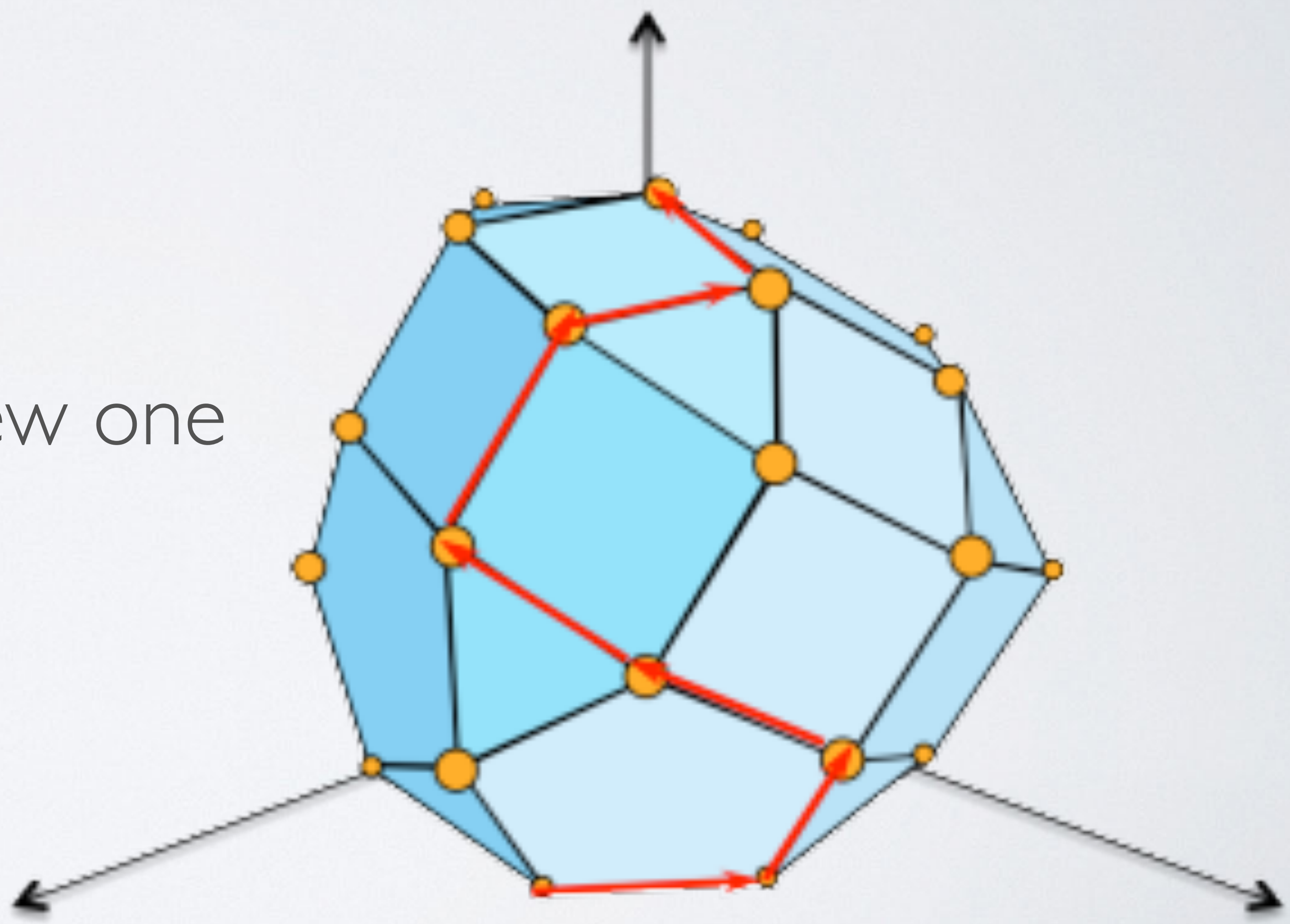
THE BIG IDEA

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

Q: How many constraints are active at each vertex?

A: N

- Pick N equations, solve them
- Drop an equation, swap in a new one
- Solve new system
- Try to decrease energy



SETUP

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} \text{maximize} \quad & t \\ \text{subject to} \quad & Ax = b \\ & t = c^T x \\ & x \geq 0 \end{aligned}$$

Simplex Tableau

$$\left(\begin{array}{cccccc|c} 1 & -c_1 & -c_2 & -c_3 & -c_4 & -c_5 & 0 \\ 0 & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & b_1 \\ 0 & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & b_2 \end{array}\right)$$

represent constraints in matrix form

STEP 1: CHOOSE BASIC SOLUTION

Basic Solution: A set of N variables that are feasible: they satisfy both equality and inequality constraints

$$\left(\begin{array}{cccccc|c} 1 & -c_1 & -c_2 & -c_3 & -c_4 & -c_5 & 0 \\ 0 & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & b_1 \\ 0 & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & b_2 \end{array}\right)$$

$\uparrow$ x_1 $\uparrow$ x_2

All non-basic variables are assumed to be zero in candidate solution

STEP 1: CHOOSE BASIC SOLUTION

$$\left(\begin{array}{cccccc|c} 1 & -c_1 & -c_2 & -c_3 & -c_4 & -c_5 & 0 \\ 0 & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & b_1 \\ 0 & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & b_2 \end{array}\right)$$

eliminate

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & -c_3 & -c_4 & -c_5 & b_0 \\ 0 & 1 & 0 & a_{13} & a_{14} & a_{15} & b_1 \\ 0 & 0 & 1 & a_{23} & a_{24} & a_{25} & b_2 \end{array}\right)$$

$x_1 = b_1$

$x_2 = b_2$

basic solution

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

STEP 2: CHOOSE ENTERING COLUMN

pick something with $c_i > 0$

this guarantees objective will increase

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & -c_3 & -c_4 & -c_5 & b_0 \\ 0 & 1 & 0 & a_{13} & a_{14} & a_{15} & b_1 \\ 0 & 0 & 1 & a_{23} & a_{24} & a_{25} & b_2 \end{array}\right)$$

x_4 → **basic solution**

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

STEP 3: CHOOSE LEAVING ROW

pick row with minimal b_i/a_{ri}

this guarantees all basic variables remain non-negative

basic solution

$$\rightarrow \left(\begin{array}{cccccc|c} 1 & 0 & 0 & -c_3 & -c_4 & -c_5 & b_0 \\ 0 & 1 & 0 & a_{13} & a_{14} & a_{15} & b_1 \\ 0 & 0 & 1 & a_{23} & a_{24} & a_{25} & b_2 \end{array}\right)$$

$$\uparrow x_4 \longrightarrow \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

STEP 4: ELIMINATE/PIVOT

pick row with minimal b_i/a_{ri}

this guarantees all basic variables remain non-negative

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & -c_3 & -c_4 & -c_5 & b_0 \\ 0 & 1 & 0 & a_{13} & a_{14} & a_{15} & b_1 \\ 0 & 0 & 1 & a_{23} & a_{24} & a_{25} & b_2 \end{array}\right)$$

$\downarrow$

$$\left(\begin{array}{cccccc|c} 1 & -c_1 & 0 & -c_3 & 0 & -c_5 & b_0 \\ 0 & a_{11} & 0 & a_{13} & 1 & a_{15} & b_1 \\ 0 & a_{21} & 1 & a_{23} & 0 & a_{25} & b_2 \end{array}\right)$$

basic solution

$$\begin{pmatrix} b_0 \\ 0 \\ b_2 \\ 0 \\ b_1 \\ 0 \end{pmatrix}$$

…rinse and repeat

PHASE 1 PROBLEM

want to solve this

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Need feasible solution: solve "phase 1" problem

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & Ax = b \\ & x \geq -t \\ & t \geq 0 \end{aligned}$$

measure infeasibility

TWO APPROACHES

Two phase approach

phase I

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & Ax = b \\ & x \geq -t \\ & t \geq 0 \end{aligned}$$

then →

phase II

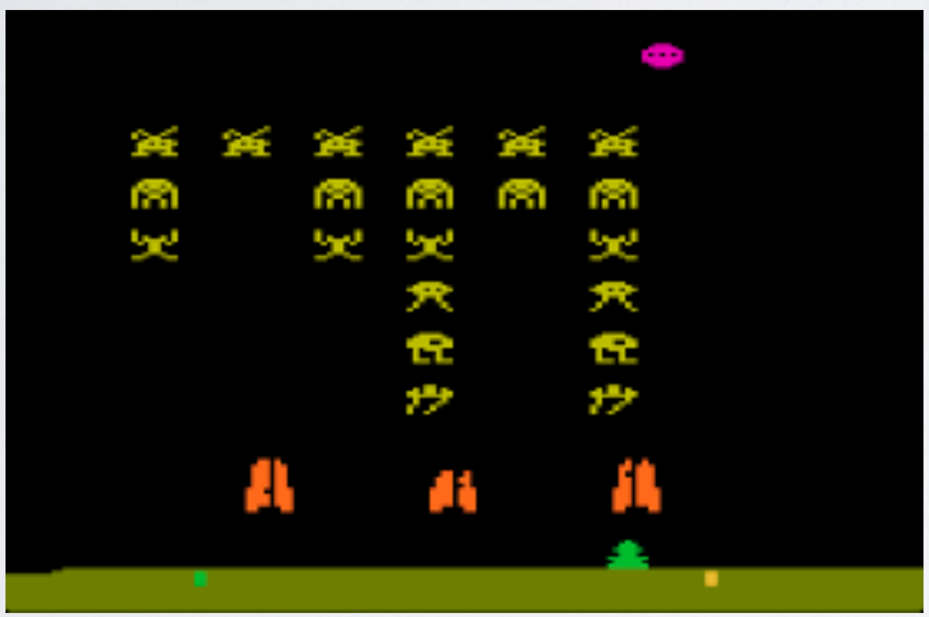
$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

"Big M" approach

$$\begin{aligned} \text{maximize} \quad & c^T x - Mt \\ \text{subject to} \quad & Ax = b \\ & x \geq -t \\ & t \geq 0 \end{aligned}$$

REINFORCEMENT LEARNING

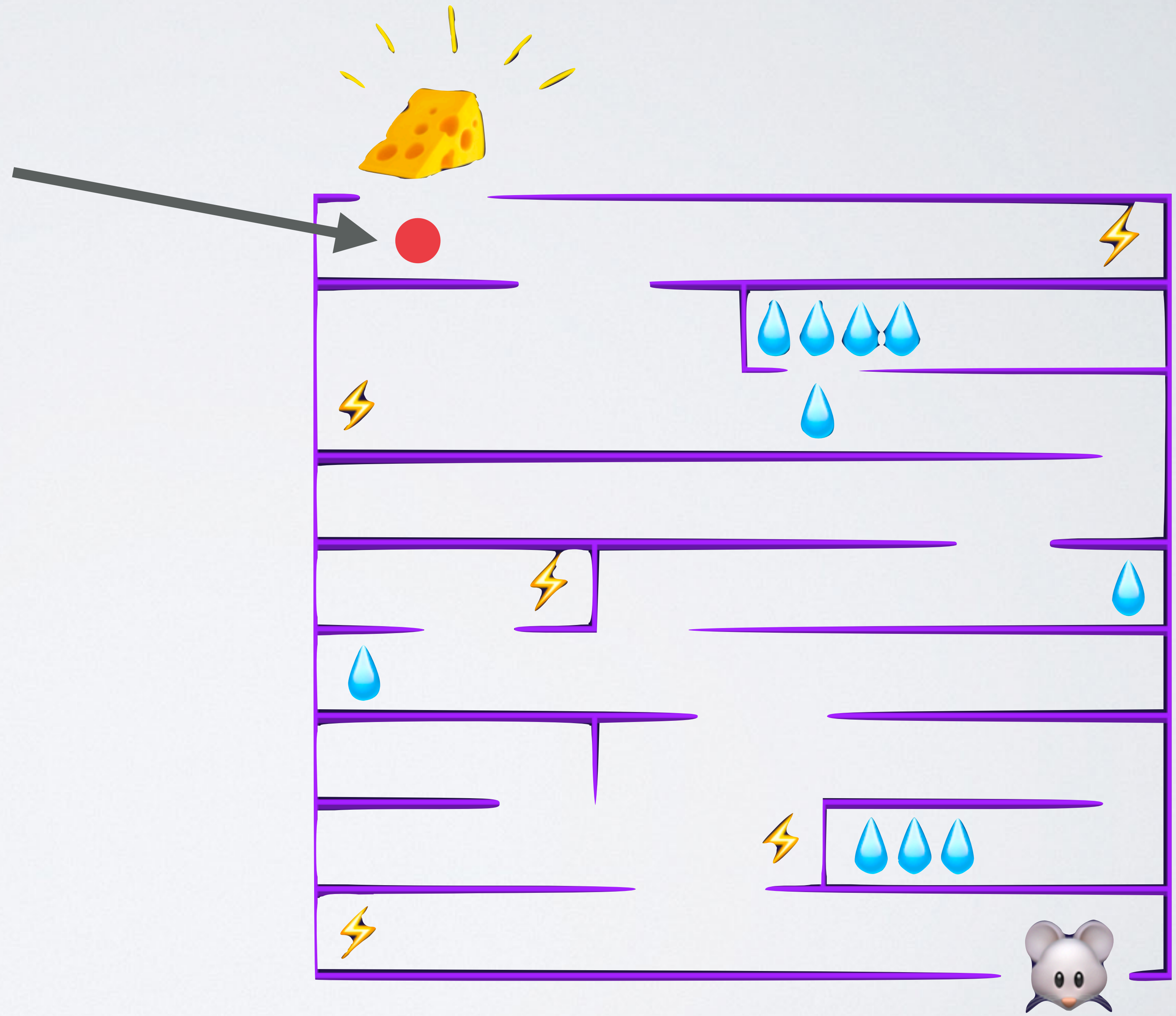
Space Invaders



Montezuma's revenge

STATE SPACE

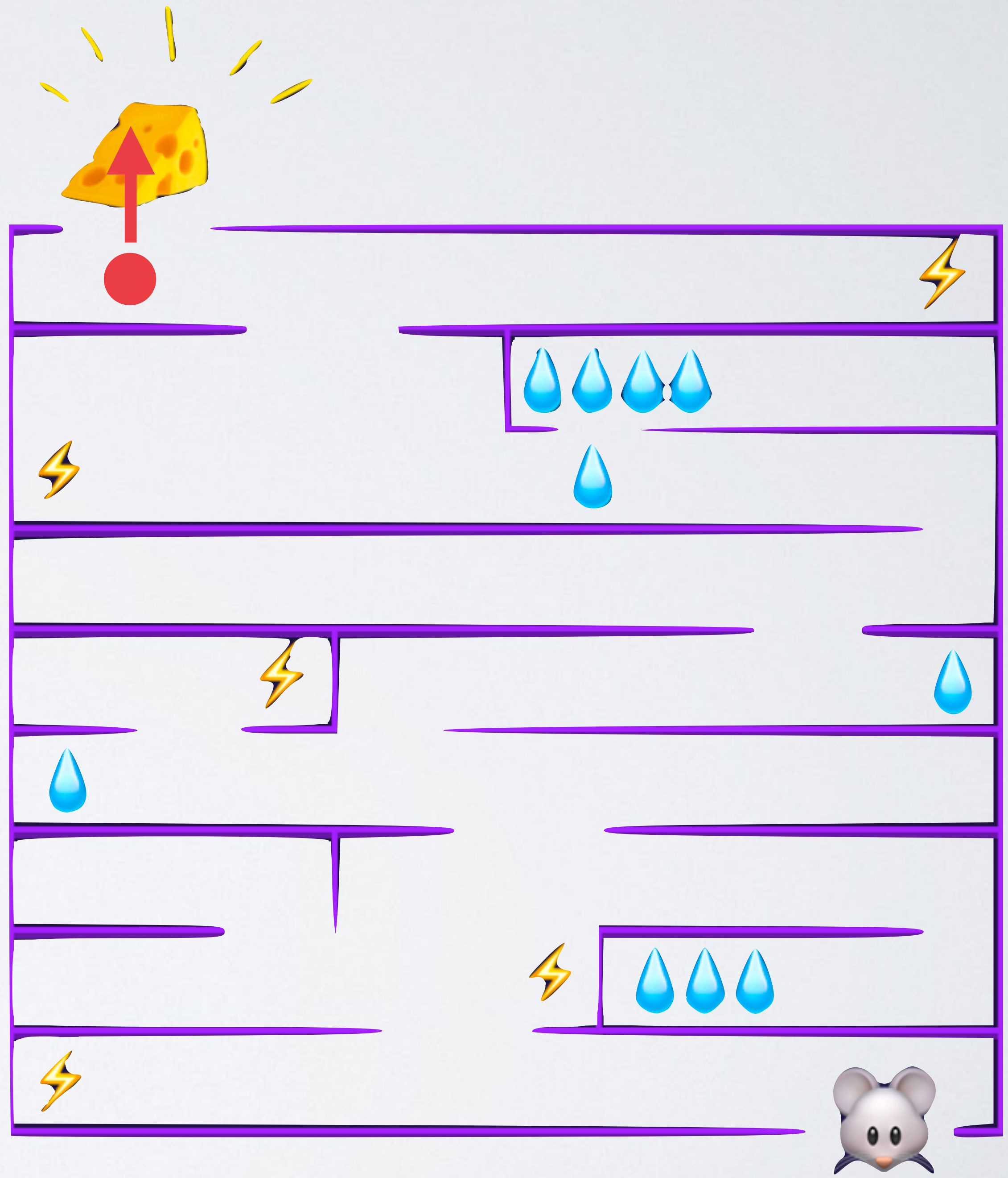
You start in a STATE



STATE SPACE

You start in a STATE

You choose an ACTION

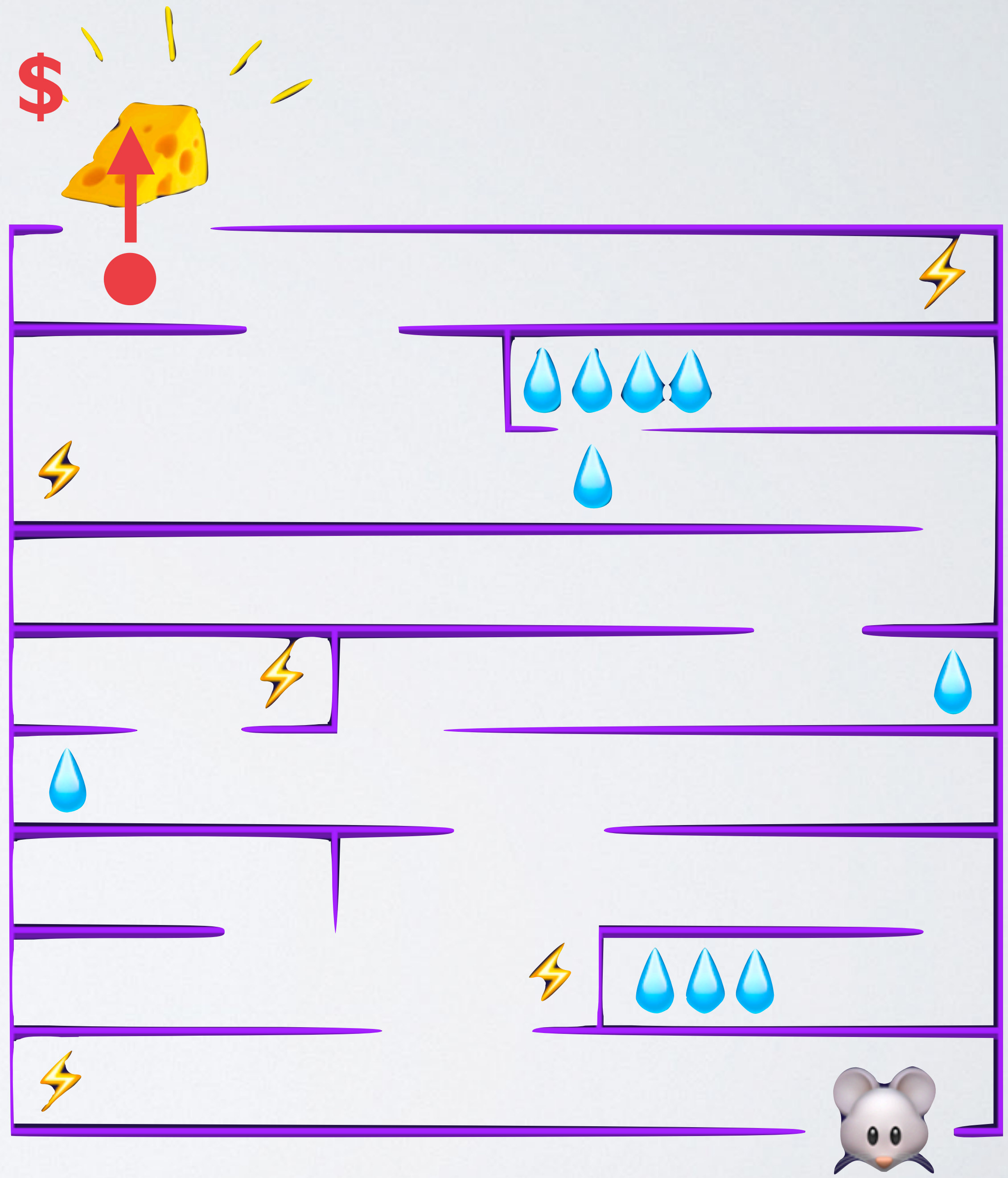


STATE SPACE

You start in a STATE

You choose an ACTION

Get a REWARD



STATE SPACE

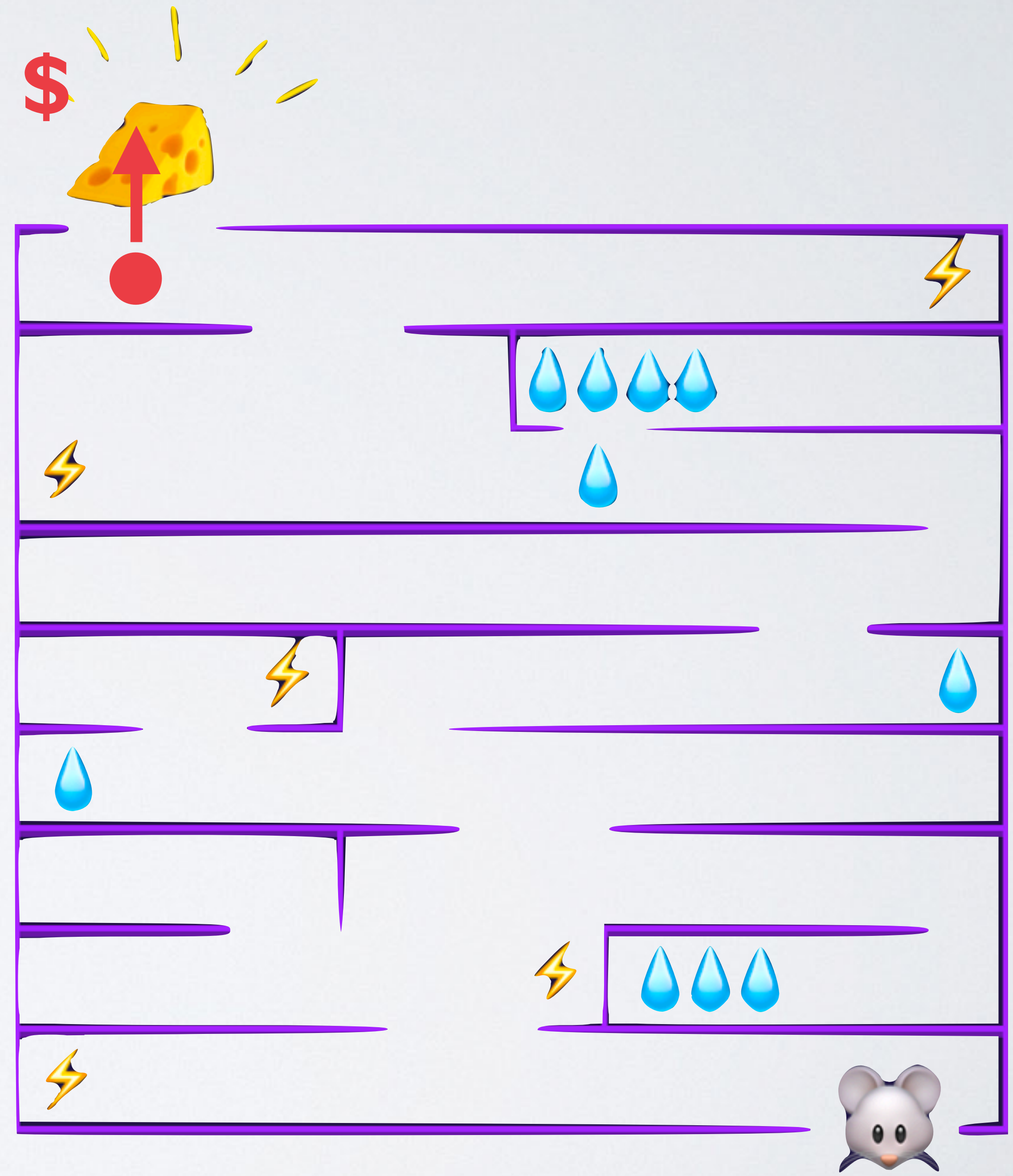
You start in a STATE

You choose an ACTION

Get a REWARD

VALUE of a state

$$V_s = \sum_{k=0}^{\infty} \gamma^k r_{s_k,a_k}$$



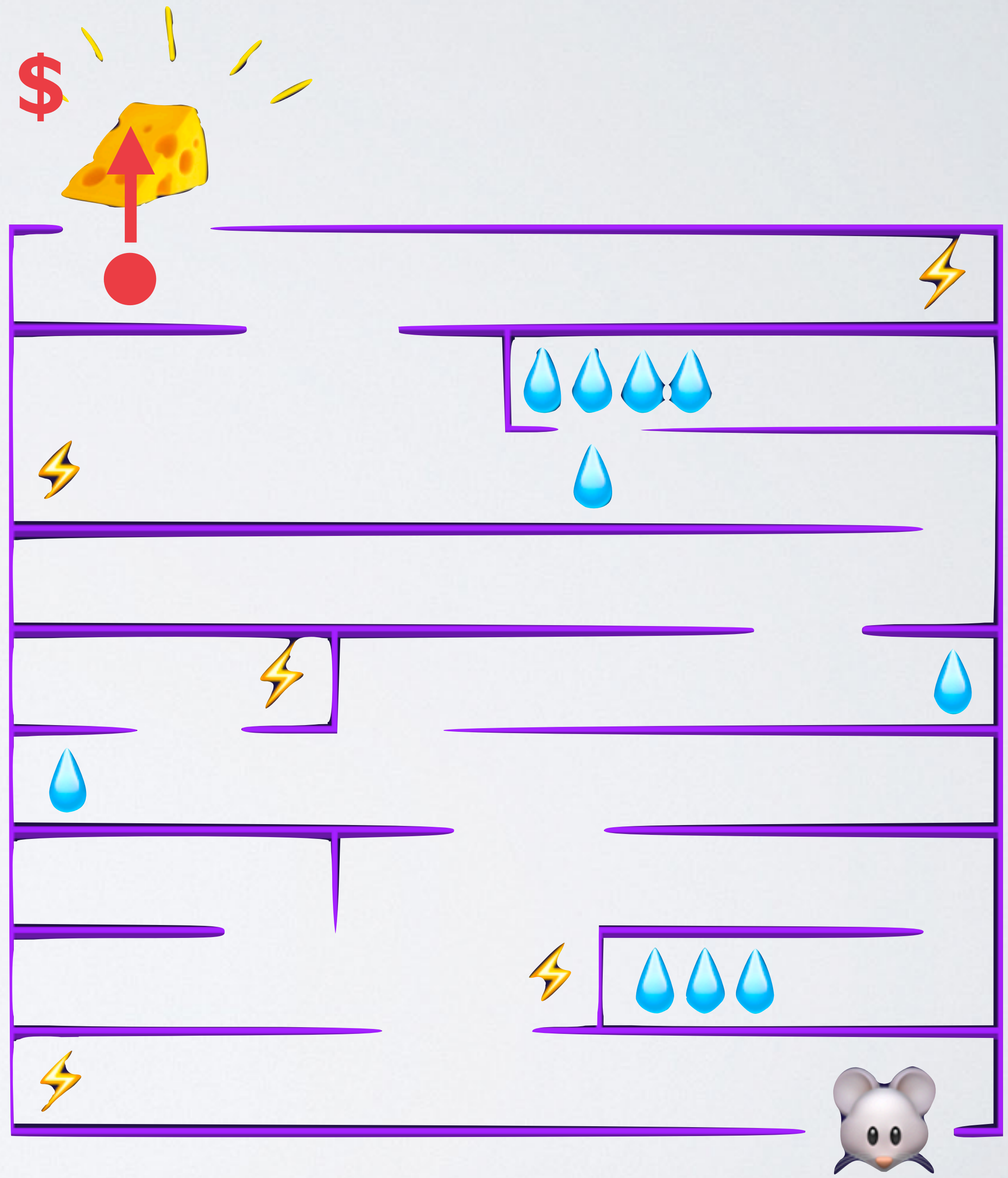
STATE SPACE

VALUE of a state

$$V_s = \sum_{k=0}^{\infty} \gamma^k r_{s_k, a_k}$$

Bellman equation:
value of *optimal* policy

$$V_s = \max_a \{ r_{s,a} + \gamma * V_{s,a} \}$$



STATE SPACE

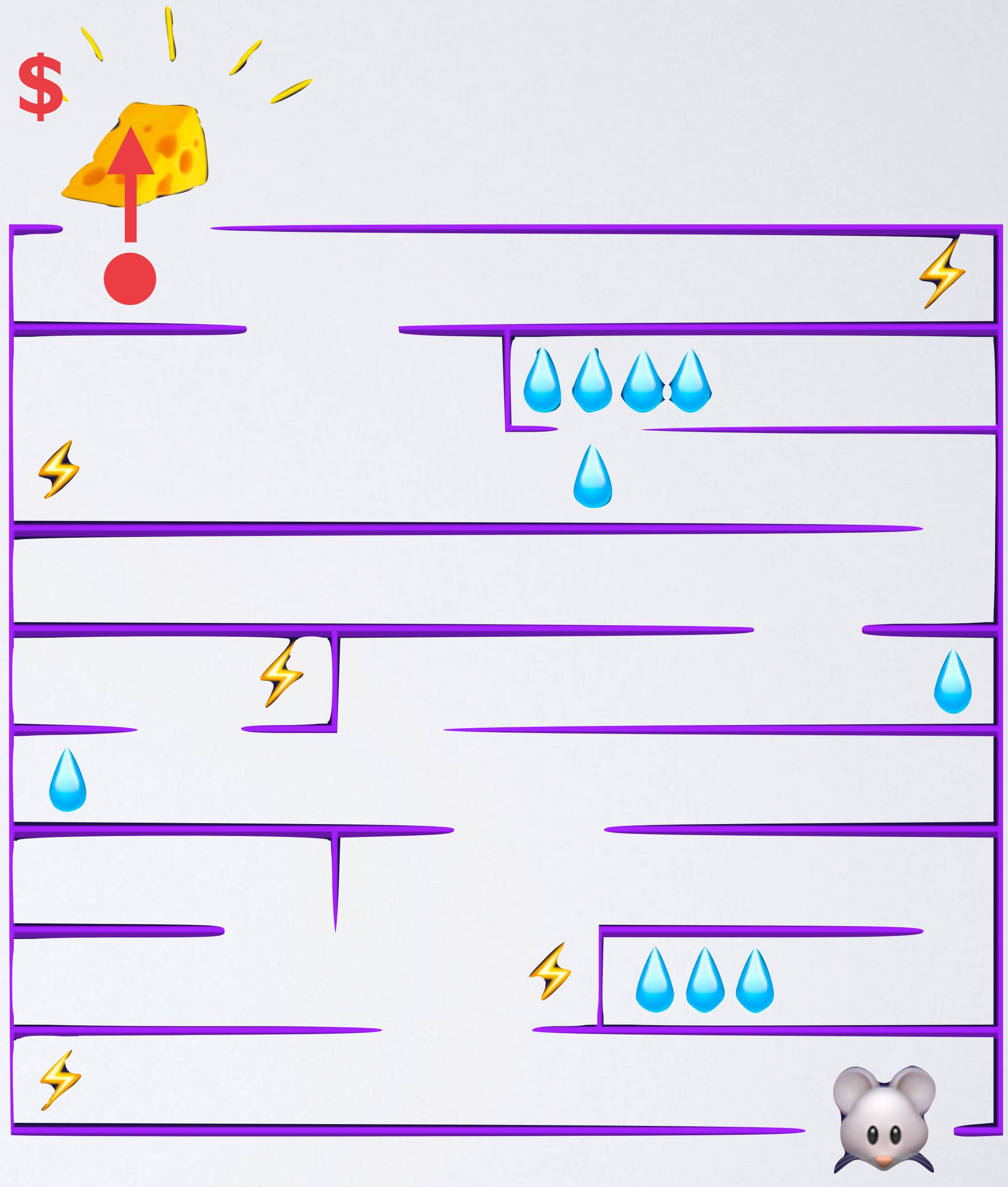
Bellman equation:
value of *optimal* policy

$$V_s = \max_a \{ r_{s,a} + \gamma * V_{s,a} \}$$

LP formulation

$$\min_v \quad \mathbf{1_n}^T v$$
$$\text{subject to}$$
$$v_s \geq r_{s,a} + \gamma V_{s,a}, \forall (s,a,r)$$

Inequality constraint
for every (s,a,r) observation



STATE SPACE

Bellman equation:
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LP formulation

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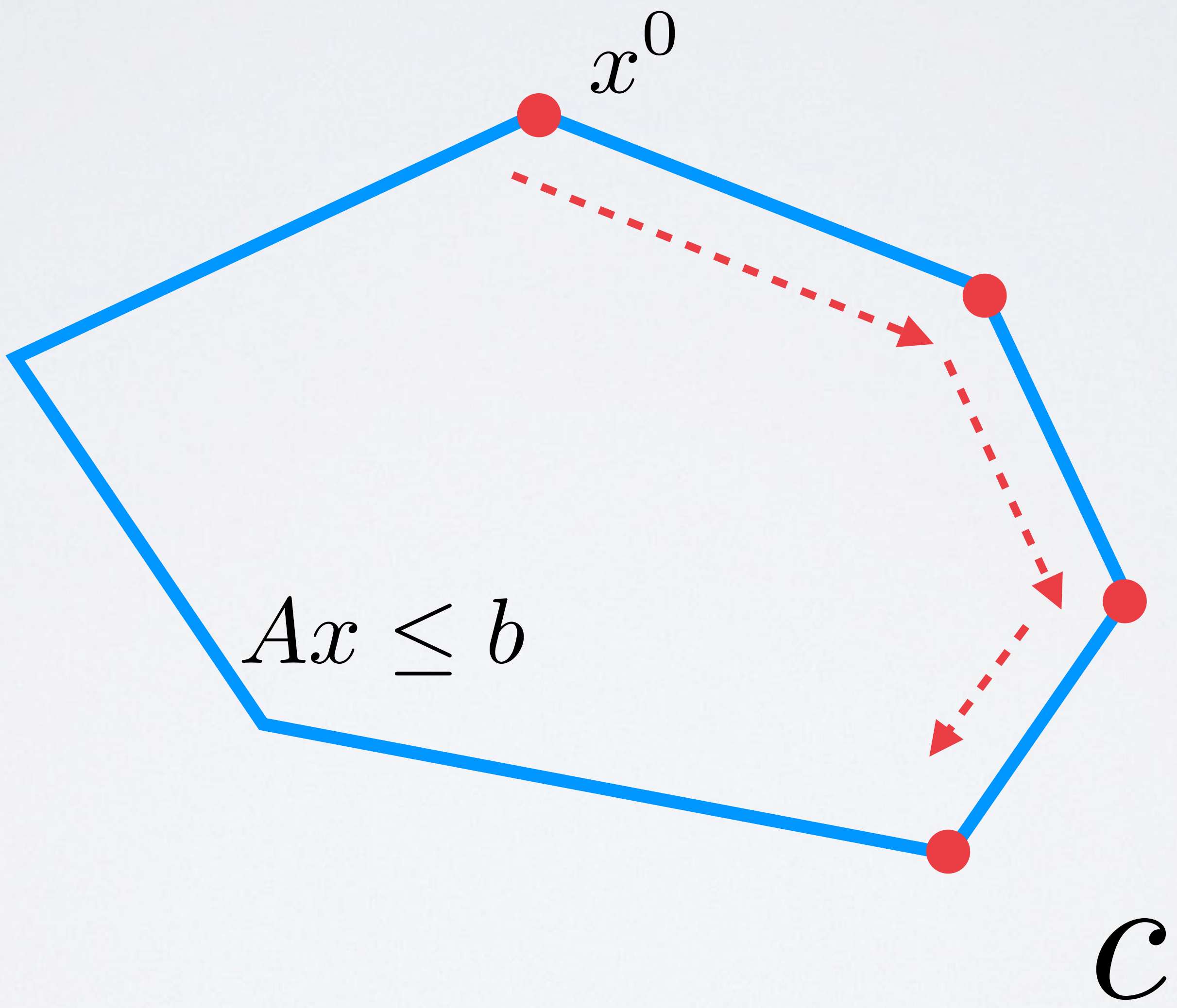
$$\text{subject to}$$

$$v_s \geq r_{s,a} + \gamma V_{s,a}, \forall (s, a, r)$$

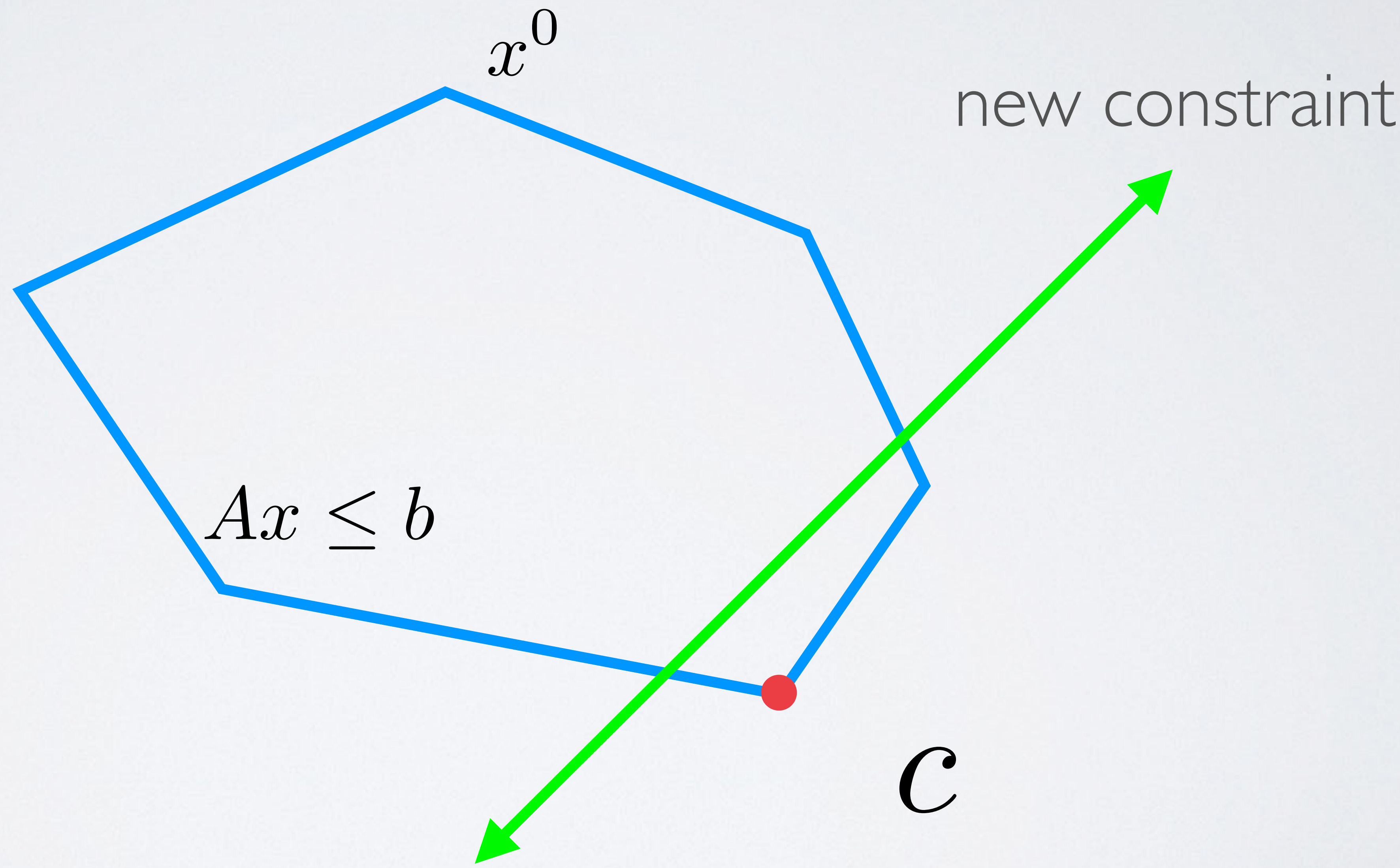
Inequality constraint
for every (s,a,r) observation

Add an equation every time
you explore the environment

MODIFYING THE PROBLEM

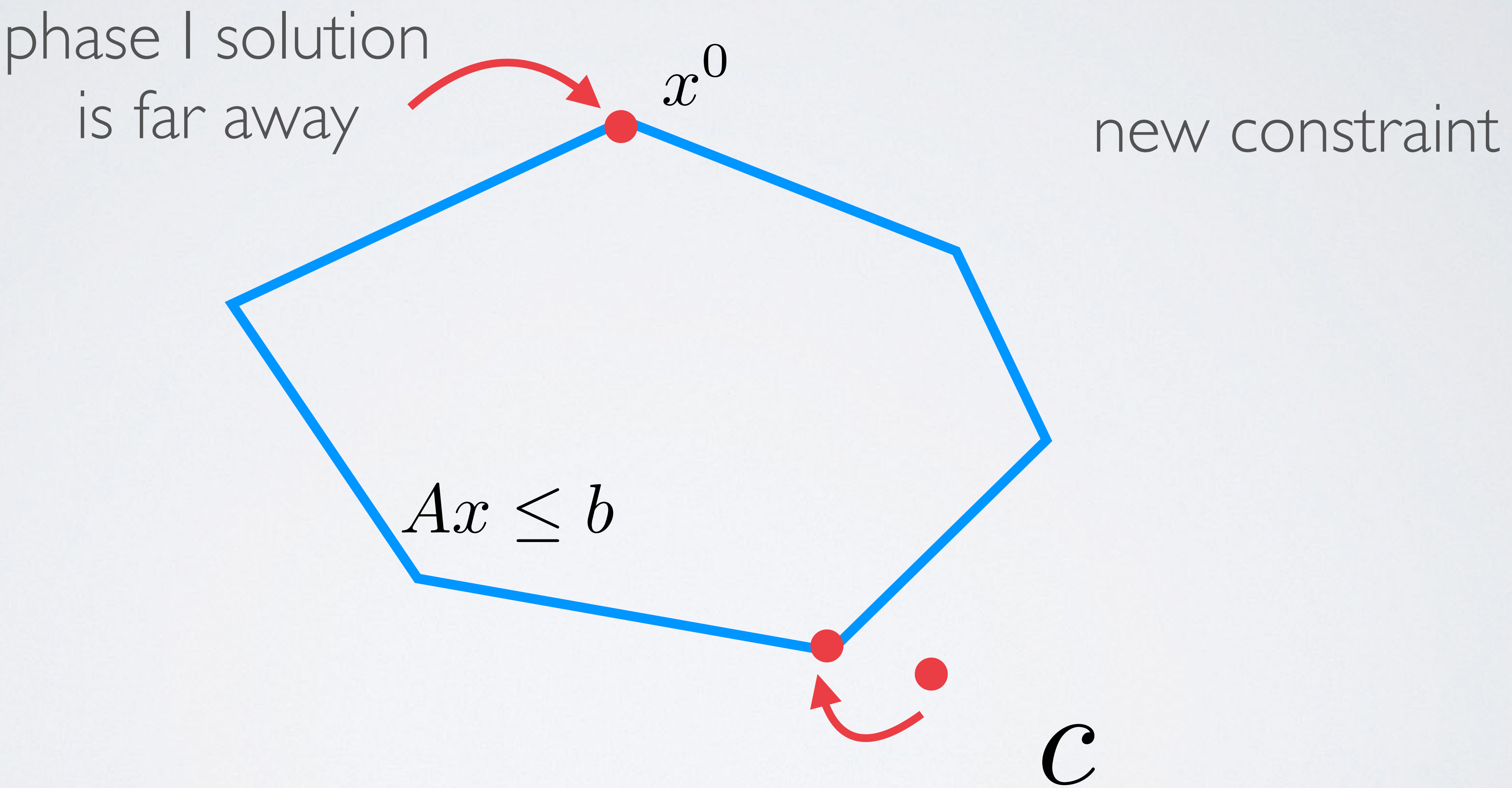


MODIFYING THE PROBLEM



Note: this happens all the time in **reinforcement learning**

MODIFYING THE PROBLEM



can't we just "snap" back onto the simplex?

DUAL SIMPLEX METHOD

primal problem

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Lagrangian

$$\max_{\lambda, \eta \leq 0} \min_x c^T x + \langle \lambda, Ax - b \rangle + \langle \eta, x \rangle$$

or

$$\max_{\lambda, \eta \leq 0} \min_x \langle \underbrace{c + A^T \lambda + \eta}_{= 0}, x \rangle - \langle \lambda, b \rangle$$

dual

$$\begin{aligned} \text{maximize} \quad & -b^T \lambda \\ \text{subject to} \quad & A^T \lambda \geq -c \end{aligned}$$

DUAL SIMPLEX METHOD

primal problem

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

dual

$$\begin{aligned} \text{maximize} \quad & -b^T \lambda \\ \text{subject to} \quad & A^T \lambda \geq -c \end{aligned}$$

Adding a constraint (row) to the primal is like adding a variable (column) to the dual!

DUAL SIMPLEX METHOD

solve dual simplex tableau

$$\left(\begin{array}{cccccc|c} 1 & -c_1 & 0 & -c_3 & 0 & -c_5 & 0 \\ 0 & a_{11} & 0 & a_{13} & 1 & a_{15} & b_1 \\ 0 & a_{21} & 1 & a_{23} & 0 & a_{25} & b_2 \end{array}\right)$$

put new row here

$$\left(\begin{array}{ccccccc|c} 1 & -c_1 & 0 & -c_3 & 0 & -c_5 & -c_6 & 0 \\ 0 & a_{11} & 0 & a_{13} & 1 & a_{15} & a_{16} & b_1 \\ 0 & a_{21} & 1 & a_{23} & 0 & a_{25} & a_{16} & b_2 \end{array}\right)$$

basic/feasible solution is still feasible

COMPLEXITY

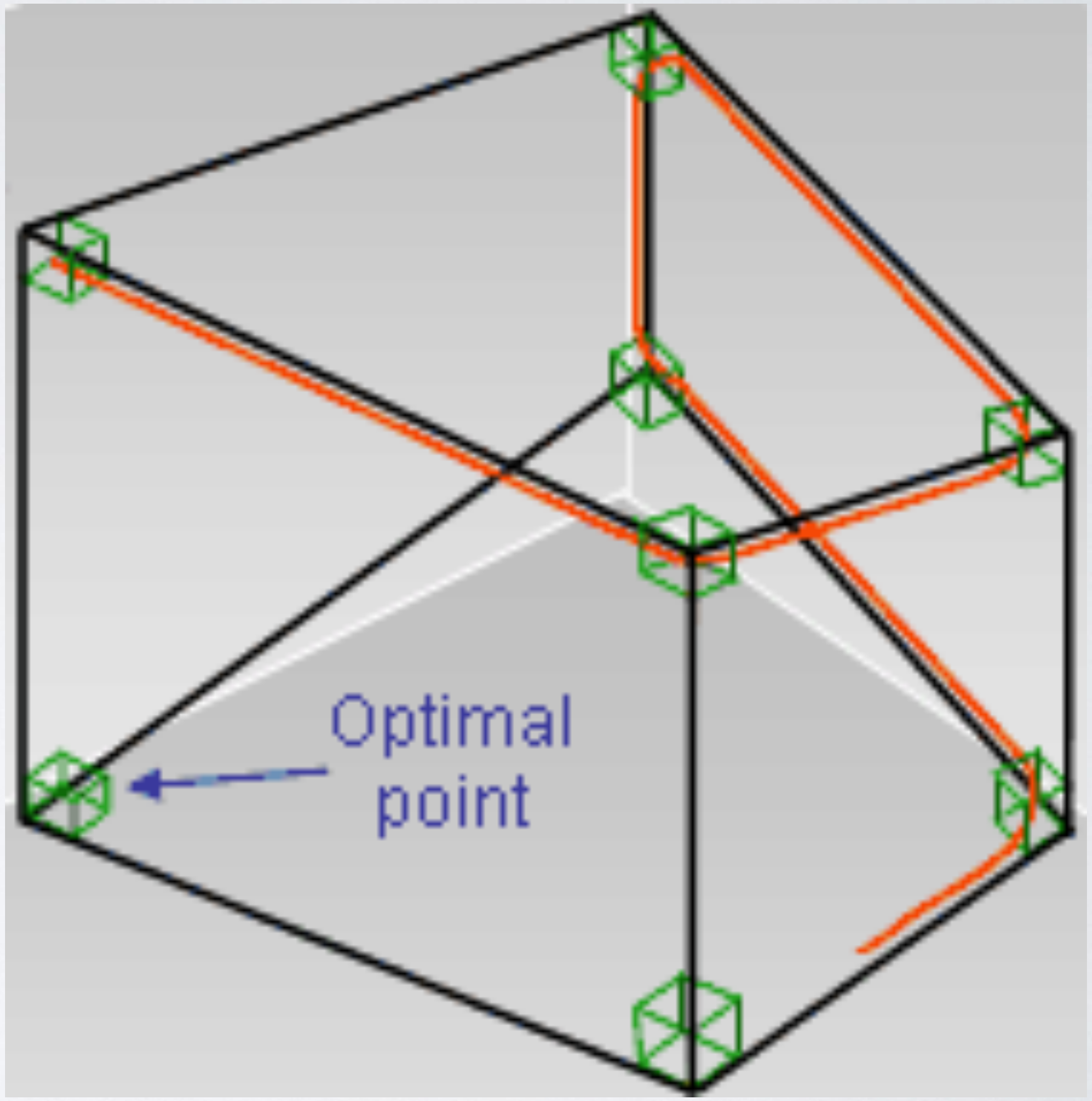
Simplex has exponential worst-case behavior

Holds for most common pivot rules

Average case complexity is polynomial for random constraints

Is there a polynomial time algorithm for LP?

Klee-Minty Cube



Klee and Minty: "How good in the Simplex algorithm?" 197