Introduction to Linear Algebra I

- Inner products
- Cauchy-Schwarz inequality
- Triangle inequality, reverse triangle inequality
- Vector and matrix norms
- Equivalence of ℓ_p norms
- Basic norm inequalities (useful for proofs)
- Matrices

Basics

- Sets, vector space
- $\mathbb{R}^N$: N-dimensional **Euclidean space**
- A **vector a** $\in \mathbb{R}^N$ is an n-tuple $[a_1, a_2, \ldots, a_N]$, where $a_i \in \mathbb{R}$.
  (think of vectors as a column vector or a $N \times 1$ matrix.)
- **Inner product:**
  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N, \langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^N a_i b_i = \mathbf{a}^T \mathbf{b} = \mathbf{b}^T \mathbf{a} = \langle \mathbf{b}, \mathbf{a} \rangle$
  (Note that $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathbb{R}$.)
- **Euclidean norm:** Induced by the inner-product
  $\mathbf{a} \in \mathbb{R}^N, \|\mathbf{a}\|_2 = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle}$
- Don't confuse the norm $\|\mathbf{x}\|_2$ with the absolute value $|x|$

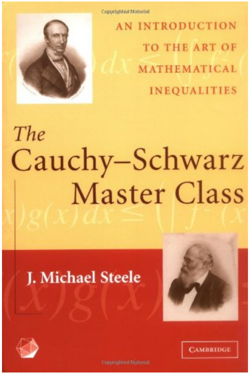
Cauchy-Schwarz inequality

Lemma (Cauchy-Schwarz inequality)

Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$,

$$|\langle \mathbf{a}, \mathbf{b} \rangle| \leq \|\mathbf{a}\|_2 \|\mathbf{b}\|_2.$$

- Probably the most important inequality out there!
- There is a book solely devoted to this inequality.
- When does it hold with equality?
- Is used to derive the triangle inequality shown next



Triangle inequality

Lemma (Triangle inequality)

Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$,

$$\|\mathbf{a} + \mathbf{b}\|_2 \leq \|\mathbf{a}\|_2 + \|\mathbf{b}\|_2.$$

- Proof uses Cauchy-Schwarz inequality (do on board)
- When does this inequality hold with equality?
- Reverse (or inverse) triangle inequalities:

$$\|\mathbf{a} + \mathbf{b}\|_2 \geq \|\mathbf{a}\|_2 - \|\mathbf{b}\|_2$$
$$\|\mathbf{a} + \mathbf{b}\|_2 \geq \|\mathbf{b}\|_2 - \|\mathbf{a}\|_2$$

What is a norm?

- Assigns a positive number to each non-zero vector
- Is only zero if the vector is an all-zero vector
- Key aspect in proving uniqueness results

Norm properties

- Homogeneity: $\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\|$, for $\mathbf{x} \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$
- Subadditivity: $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$
- Separability: If and only if $\|\mathbf{x}\| = 0$, then $\mathbf{x} = 0$

It's time to play "IS IT A NORM?!?"

- $\|\mathbf{x}\|_2$
- $\|\mathbf{x}\|_0$ counts the number of non-zeros in $\mathbf{x}$
- $\|\mathbf{x}\|_1$ or $|\mathbf{x}|$
- $\|\nabla \mathbf{x}\|_2$
- $\sqrt{\mathbf{x}^T A \mathbf{x}}$, for some matrix A.

ℓ_p norms

Definition (ℓ_p norms)

$$p \geq 1, \mathbf{a} \in \mathbb{R}^N, \|\mathbf{a}\|_p = \left(\sum_{i=1}^{N} |a_i|^p \right)^{1/p}$$

- ℓ_2 norm: $p = 2, \|\mathbf{a}\|_2 = \sqrt{\sum_i |a_i|^2}$
- ℓ_1 norm: $p = 1, \|\mathbf{a}\|_1 = \sum_i |a_i|$
- ℓ_∞ norm: $p = \infty, \|\mathbf{a}\|_\infty = \max_i |a_i|$

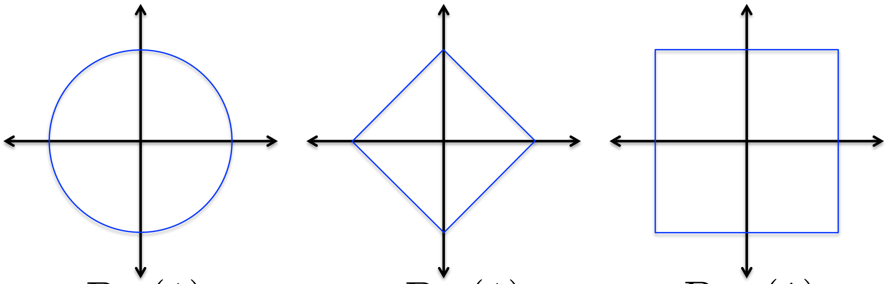
Lemma (Minkowski's inequality)

$$1 \leq p \leq \infty, \quad \|\mathbf{a} + \mathbf{b}\|_p \leq \|\mathbf{a}\|_p + \|\mathbf{b}\|_p$$

ℓ_p-norm balls

Definition (ℓ_p ball)

$$\epsilon \geq 0, \quad B_{\ell_p}(\epsilon) = B_p(\epsilon) = \{\mathbf{a} \mid \|\mathbf{a}\|_p \leq \epsilon\}$$



$B_{\ell_2}(1)$ $B_{\ell_1}(1)$ $B_{\ell_\infty}(1)$

$B_p(1)$ is referred to as the *unit ball* (i.e, $\epsilon = 1$) .

Equivalence of norms

Given any two norms, say ℓ_p and ℓ_q, $\exists\ \alpha, \beta > 0$ such that

$$\forall \mathbf{a} \in \mathbb{R}^N,\ \alpha \|\mathbf{a}\|_q \leq \|\mathbf{a}\|_p \leq \beta \|\mathbf{a}\|_q.$$

- $\|\mathbf{a}\|_\infty \leq \|\mathbf{a}\|_2 \leq \sqrt{N} \|\mathbf{a}\|_\infty$
- $\|\mathbf{a}\|_\infty \leq \|\mathbf{a}\|_1 \leq N \|\mathbf{a}\|_\infty$
- $\|\mathbf{a}\|_2 \leq \|\mathbf{a}\|_1 \leq \sqrt{N} \|\mathbf{a}\|_2$

- This implies that all p-norms behave—at least in principle—similarly
- However, we will show that they have very distinct properties

Lemma (General equivalence of ℓ_p norms)

$$1 \leq p < q, \quad \|\mathbf{a}\|_q \leq \|\mathbf{a}\|_p \leq N^{1/p - 1/q} \|\mathbf{a}\|_q$$

Two important inequalities

Hölder's inequality:

- $|\langle \mathbf{a}, \mathbf{b} \rangle| \leq \|\mathbf{a}\|_p \|\mathbf{b}\|_q$ with $1/p + 1/q = 1$ and $p, q \in [1, \infty]$
- p and q are so-called dual norms
- Generalization of the Cauchy-Schwarz inequality

Jensen's inequality:

- Let $f(x)$ be a convex function with $x_1, x_2 \in \mathbb{R}$ and for $t \in [0, 1]$

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

- Also
$$f\left(\frac{\sum_i a_i x_i}{\sum_i a_i}\right) \leq \frac{\sum_i a_i f(x_i)}{\sum_i a_i}$$

Collection of vectors, Subspaces

A set of T-vectors, $V = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_T\}$

- **Linear combination:** $\sum_{k=1}^{T} \alpha_k \mathbf{a}_k, \alpha_k \in \mathbb{R}$
- **Linearly independent:** No vector in V can be written as linear combination of others
- **Span:** $\mathrm{Span}(V) = \{\mathbf{x} \,|\, \mathbf{x} = \sum_k \alpha_k \mathbf{a}_k, \alpha_k \in \mathbb{R}\}$

Definition (Subspace)

A collection of vectors $V \subset \mathbb{R}^N$ is a subspace iff it is closed under linear combinations

$$\mathbf{a}, \mathbf{b} \in V \implies \alpha \mathbf{a} + \beta \mathbf{b} \in V, \alpha, \beta \in \mathbb{R}$$

- **Basis of a subspace:** A linearly independent *spanning* set
- **Dimensionality of a subspace:** #elements in a basis

Matrix

- $A \in \mathbb{R}^{M \times N}$: A matrix of dimension $M \times N$
- $A = [a_{ij}] = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N], \mathbf{a}_i \in \mathbb{R}^M$
- $\mathrm{rank}(A)$ = largest number of linearly *independent* columns
- $\mathrm{rank}(A) = \mathrm{rank}(A^T) \leq \min(M, N)$
- A is *full-rank* if $\mathrm{rank}(A) = \min(M, N)$.

Matrices are *representations of linear operators.*

$A : \mathbb{R}^N \to \mathbb{R}^M$
$\mathbf{x} \in \mathbb{R}^N \mapsto A\mathbf{x} \in \mathbb{R}^M$

Examples of linear operators that aren't matrices?

Matrix norms

Definition (Spectral norm)

$$\|A\|_{2,2} = \max_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \max_{\|\mathbf{x}\|_2 = 1} \|A\mathbf{x}\|_2$$

- The norm used above is the *induced* norm or the ℓ_2-norm.
- Quantifies the maximum increase in length of unit-norm vectors due to the operation of the matrix A
- $\|A\|_{2,2}$ is equal to the largest *singular value* of A (more on this later)
- $\|A\mathbf{x}\|_2 \leq \|A\|_{2,2}\|\mathbf{x}\|_2$ (Question: When is it equal?)

Lemma

$$\|AB\|_{2,2} \leq \|A\|_{2,2}\|B\|_{2,2}$$

- Can you show this?

Induced matrix norms

Definition

$$\|A\|_{p,q} = \max_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_q}{\|\mathbf{x}\|_p} = \max_{\|\mathbf{x}\|_p=1} \|A\mathbf{x}\|_q$$

- $\|A\|_{2,2}$ the maximum singular value of A
- $\|A\|_{1,1}$: maximum of the absolute column sums
- $\|A\|_{\infty,\infty}$: maximum of the absolute row sums

- $\|A\mathbf{x}\|_q \leq \|A\|_{p,q}\|\mathbf{x}\|_p$ (by definition)
- $\|A\|_{2,2}^2 \leq \|A\|_{1,1}\|A\|_{\infty,\infty}$ (similar to Hölder's inequality)

Note: We get lazy and write $\|A\|_2$ for $\|A\|_{2,2}$

Other frequently-used matrix norms

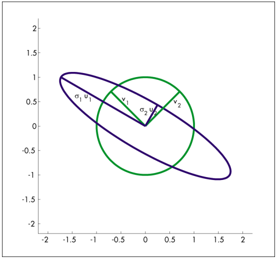
- Frobenius norm:
  - Definition: $\|A\|_F = \sqrt{\sum_{i,j} |A_{i,j}|^2}$
  - Alternative definition: $\|A\|_F = \sqrt{\text{trace}(A^T A)} = \sqrt{\text{trace}(AA^T)}$
  - The Frobenius norm is not an induced norm
- Nuclear norm:
  - Definition: $\|A\|_* = \text{trace}(\sqrt{A^T A}) = \sum_{i=1}^{\min\{M,N\}} \sigma_i$
  - With σ_i being the singular values of the matrix A
  - The nuclear norm is not an induced norm
- ALL matrix norms are also equivalent $\rightarrow$ Wikipedia

Eigenvectors and eigenvalues

- Let A be a $N \times N$ square matrix
- $\mathbf{x}$ is an eigenvector and λ is an eigenvalue of A is

$$A\mathbf{x} = \lambda\mathbf{x}$$

- **Intuition:** eigenvectors are vectors in $\mathbb{R}^N$ whose direction is preserved under action of A; however, length may change
- Eigen-decomposition: $A = UDU^{-1}$

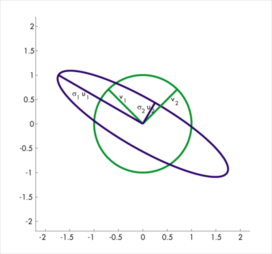


Spectral Theorem

Theorem

If $A = A^H$, then

- *The matrix is "symmetric"*
- *all eigenvalues are real*
- *eigenvectors with different eigenvalues are perpendicular*
- *there exists a complete orthogonal basis of eigenvectors.*



Singular value decomposition (SVD)

Definition (SVD)

Any matrix $A \in \mathbb{R}^{M \times N}$ can be written as

$$A = U \Sigma V^T,$$

where $U \in \mathbb{R}^{M \times M}$ and $V \in \mathbb{R}^{N \times N}$ are unitary and $\Sigma \in \mathbb{R}^{M \times N}$ is diagonal.

- Diagonal entries of $\Sigma = \{\sigma_i\}$ are called the singular values; they are positive dand real. Typically, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$
- Singular values are the eigenvalues of $\sqrt{A^T A}$ and $\sqrt{A A^T}$.
- If $A = A^T$, singular values are same as the eigenvalues
- Geometric picture and other properties, read Wikipedia
- Very useful matrix decomposition!

Singular value decomposition (SVD)

Definition (SVD)

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- If A^{-1} exists, then $A^{-1} = V\Sigma^{-1}U^T$.
- Even if A is singular, we can define a *pseudo-inverse* $A^\dagger$ as follows:

$$A^\dagger = V\widehat{\Sigma}^{-1}U^T,$$

  where $\widehat{\Sigma}^{-1}$ has the diagonal terms $1/\sigma_i$ if $\sigma_i \neq 0$, and zero otherwise
- The ratio of the largest to smallest singular value is the so-called *condition number* of A

Solving $\mathbf{y} = A\mathbf{x}$ (square case)

Scenario: A is **full-rank**, $M = N$ (square matrix)
(full rank implies that A^{-1} exists)

Given $\mathbf{y}$, the *unique* solution $\mathbf{x}$ is

$$\widehat{\mathbf{x}} = A^{-1}\mathbf{y}$$

Geometric picture: A is a *one-to-one*, *onto* map from $\mathbb{R}^N$ to $\mathbb{R}^M = \mathbb{R}^N$

Block Inversion Formulas

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix}$$

We can solve using elimination...

$$\begin{pmatrix} I & A^{-1}B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix}$$

$$\begin{pmatrix} I & A^{-1}B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^{-1}b \\ c \end{pmatrix}$$

$$\begin{pmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^{-1}b \\ c - CA^{-1}b \end{pmatrix}$$

$$y = (D - CA^{-1}B)^{-1}(c - CA^{-1}b)$$

The Schur Complement

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix}$$

"You Schur look great today!"

$$S = (D - CA^{-1}B)$$

The Schur complement

- S^{-1} is a diagonal entry in the matrix inverse
- The block matrix is invertible iff S is invertible
- Block matrix is PSD iff A, S are PSD