CONVOLUTIONS AND THE FFT

Lecture 2 - CMSC764

LINEAR FILTER

$$x \circ f[k] = \sum_i x[k+i] f[i]$$

Filter/stencil	$[-1$	1	$0]$		
Signal	2	3	1	1	4
Output		1			

LINEAR FILTER

$$x \circ f[k] = \sum_i x[k+i] f[i]$$

Filter/stencil	$[-1 \quad 1 \quad 0]$
Signal	$2 \quad 3 \quad 1 \quad 1 \quad 4$
Output	$1 \ -2$

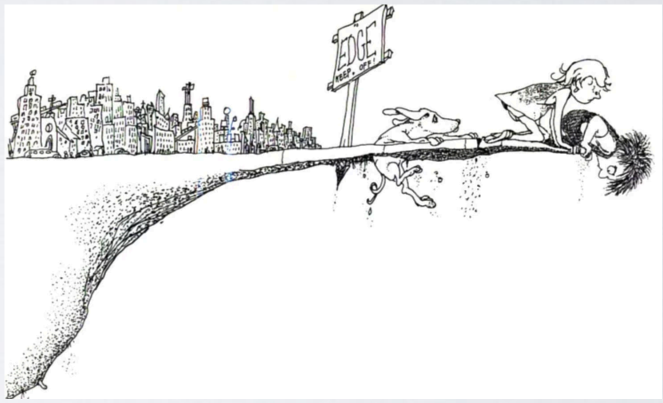
LINEAR FILTER

$$x \circ f[k] = \sum_i x[k+i] f[i]$$

Filter/stencil $\quad [-1 \quad 1 \quad 0]$

Signal $\quad 2 \quad 3 \quad 1 \quad 1 \quad 4$

Output $\quad 1 \quad -2 \quad 0$



Now what??

CIRCULANT BOUNDARY

Filter/stencil $0 \qquad [-1 \quad 1 \quad 0]$

Signal $2 \quad 3 \quad 1 \quad 1 \quad 4 \quad 2$

Output $1 \quad -2 \quad 0 \quad 3$

End of signal

CIRCULANT BOUNDARY

Filter/stencil	1	0			$[-1$	1	$0]$
Signal	2	3	1	1	4	2	3
Output	-2	1	-2	0	3	-2	

End of signal

LINEAR FILTER

Filter/stencil $[-1 \quad 1 \quad 0]$

$$x \circ f[k] = \sum_i x[k+i] f[i]$$

Filter/stencil $[-1 \quad 1 \quad 0]$

Signal $2 \quad 3 \quad 1 \quad 1 \quad 4$

Output $1 \; -2$

LINEAR **CONVOLUTION**

Filter/stencil $[-1 \quad 1 \quad 0]$

$$x \circ f[k] = \sum_i x[k+i] f[-i]$$

Lookie Here!

Filter/stencil $[0 \quad 1 \quad -1]$

Signal $2 \quad 3 \quad 1 \quad 1 \quad 4$

Output $2 \quad 0$

CONVOLUTION MATRIX

Filter/stencil $[0 \quad 1 \quad -1]$

Signal $2 \quad 3 \quad 1 \quad 1 \quad 4$

Output $2 \quad 0$

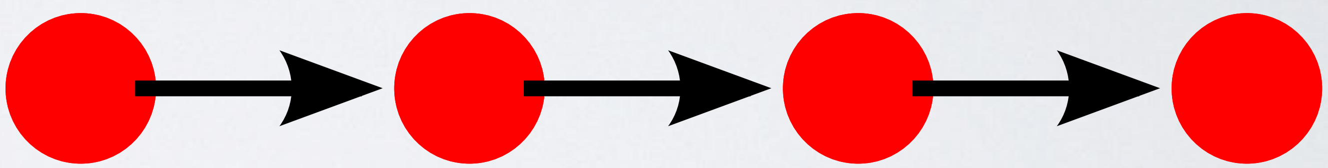
Conv matrix Signal

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \\ 4 \end{pmatrix}$$

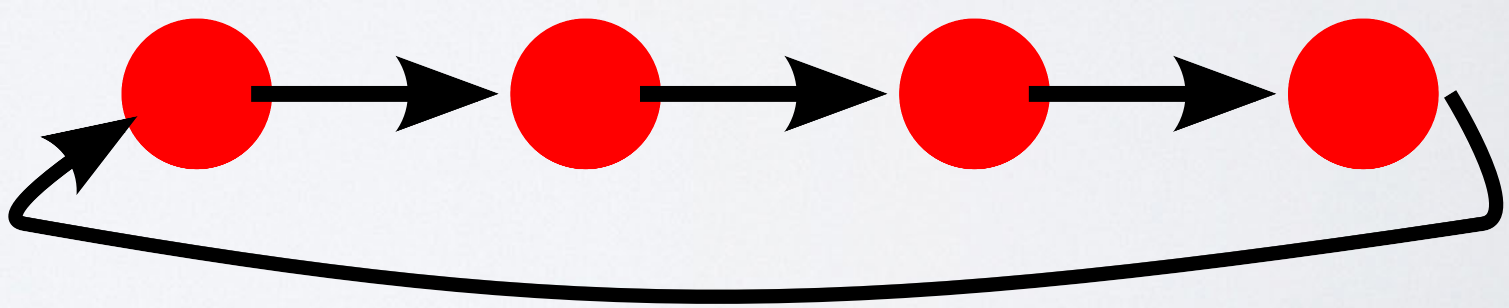
TWO COMMON USES FOR CONVOLUTIONS

DISCRETE GRADIENT

Neumann
$$\begin{pmatrix} u_1 - u_0 \\ u_2 - u_1 \\ u_3 - u_2 \end{pmatrix}$$

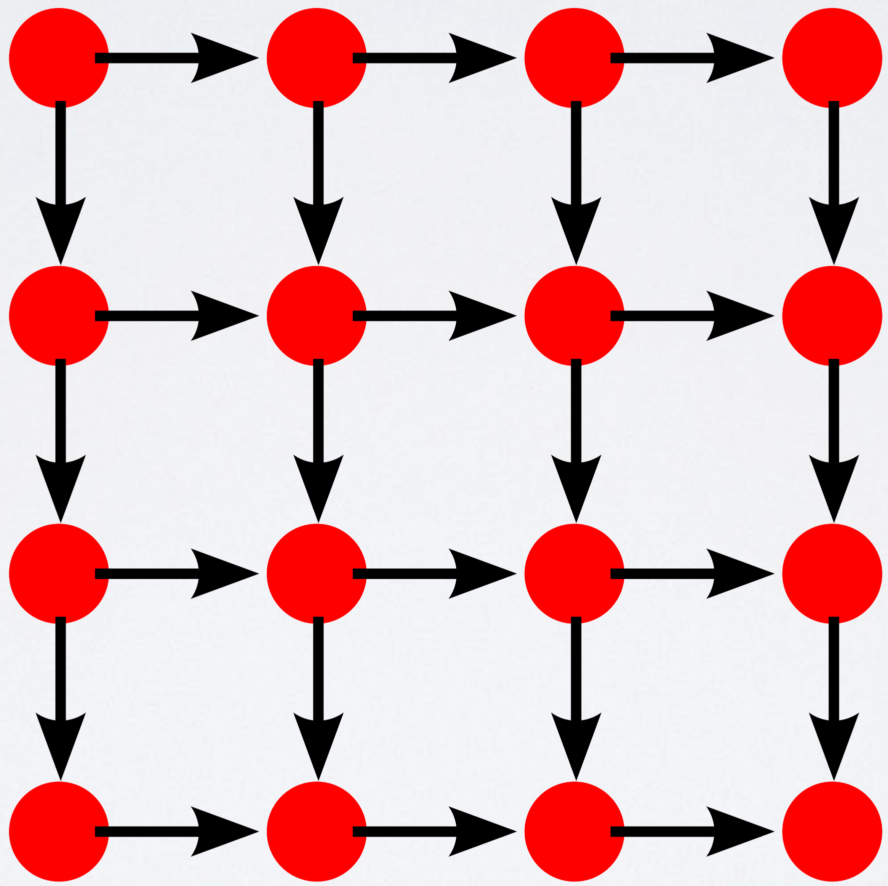


Circulant
$$\begin{pmatrix} u_1 - u_0 \\ u_2 - u_1 \\ u_3 - u_2 \\ {\color{red} u_0 - u_3} \end{pmatrix}$$



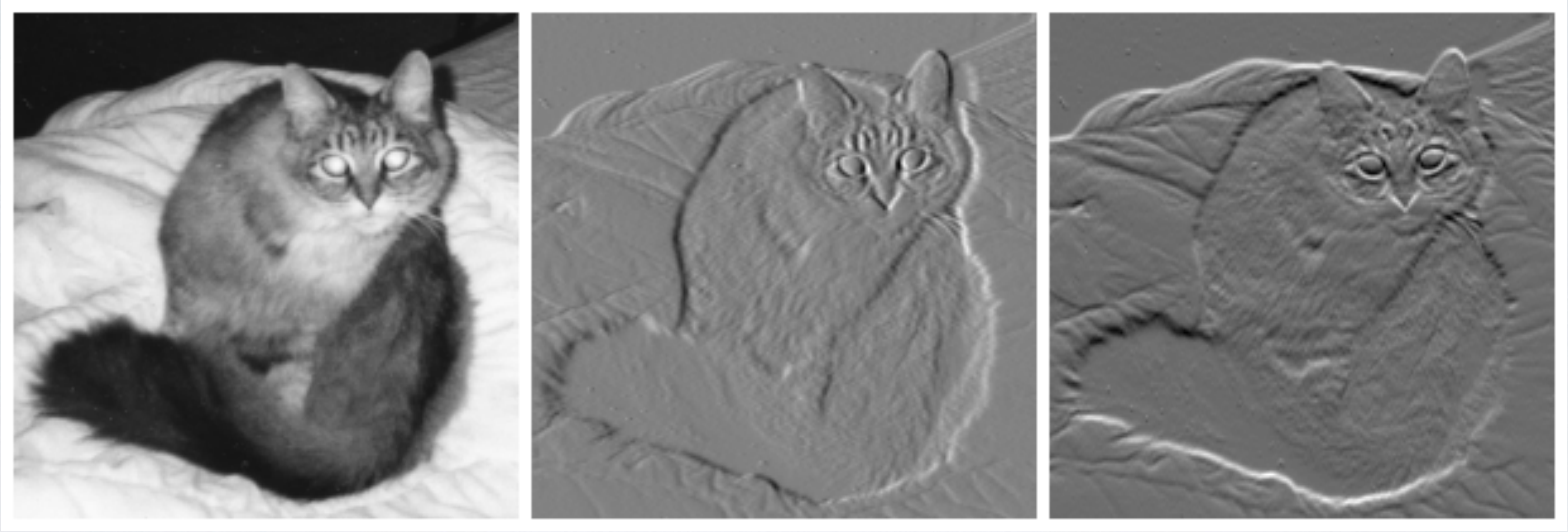
DISCRETE GRADIENT IN 2D

$$(\nabla x)_{ij} = (x_{i+1,j} - x_{i,j}, x_{i,j+1} - x_{i,j})$$



TV IN 2D

$$(\nabla x)_{ij} = (x_{i+1,j} - x_{i,j}, x_{i,j+1} - x_{i,j})$$



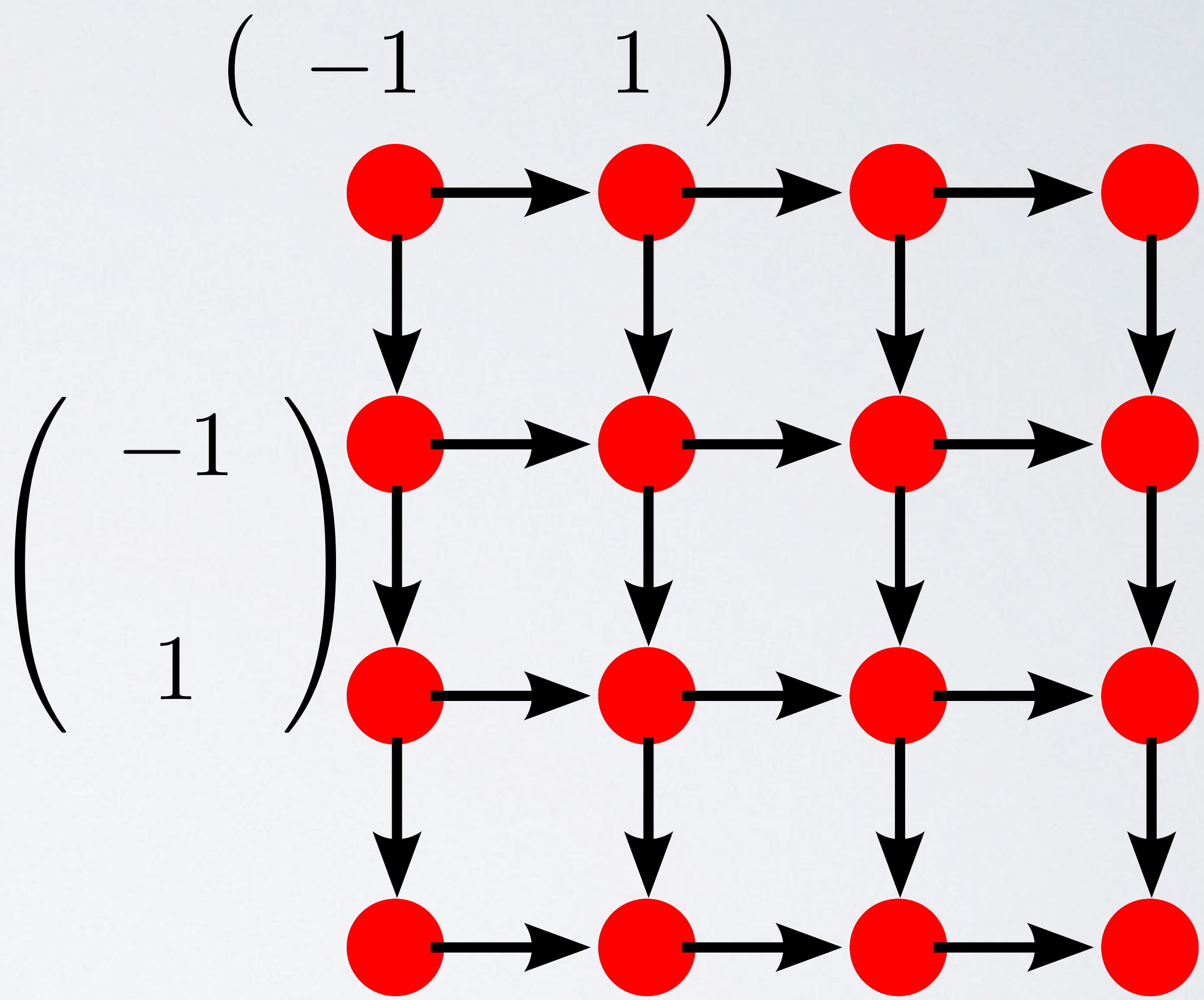
I $\qquad$ $g_x = D_x I$ $\qquad$ $g_y = D_y I$

GRADIENT ON IMAGES

- Two linear filters
- x-stencil = (-1 1 0)
- y-stencil = (-1 1 0)’

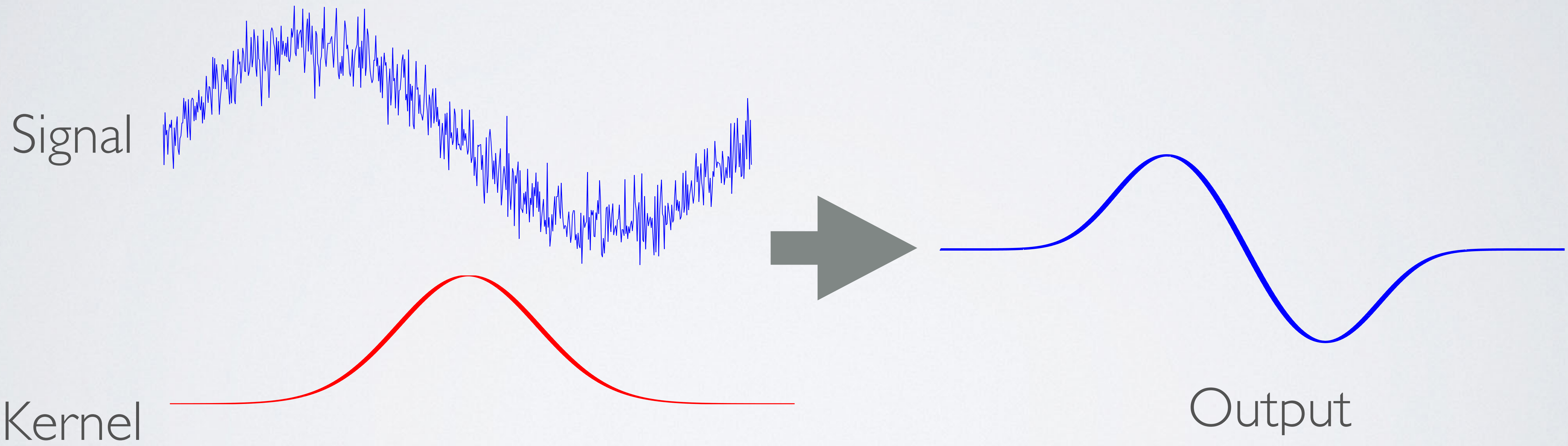
…or…

- Two linear convolutions
- x-kernel = (0 1 -1)
- y-kernel = (0 1 -1)’

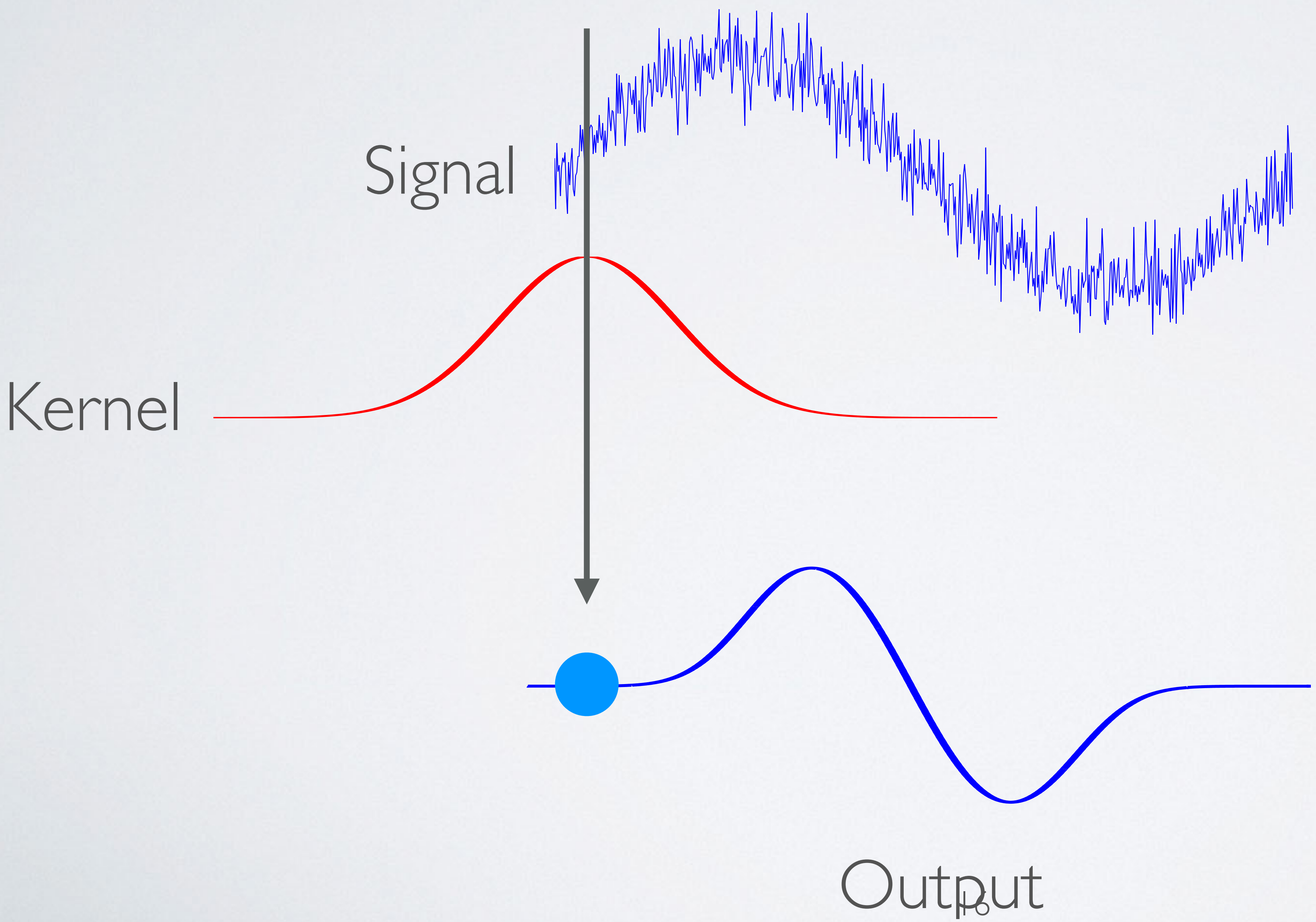


GAUSSIAN FILTER

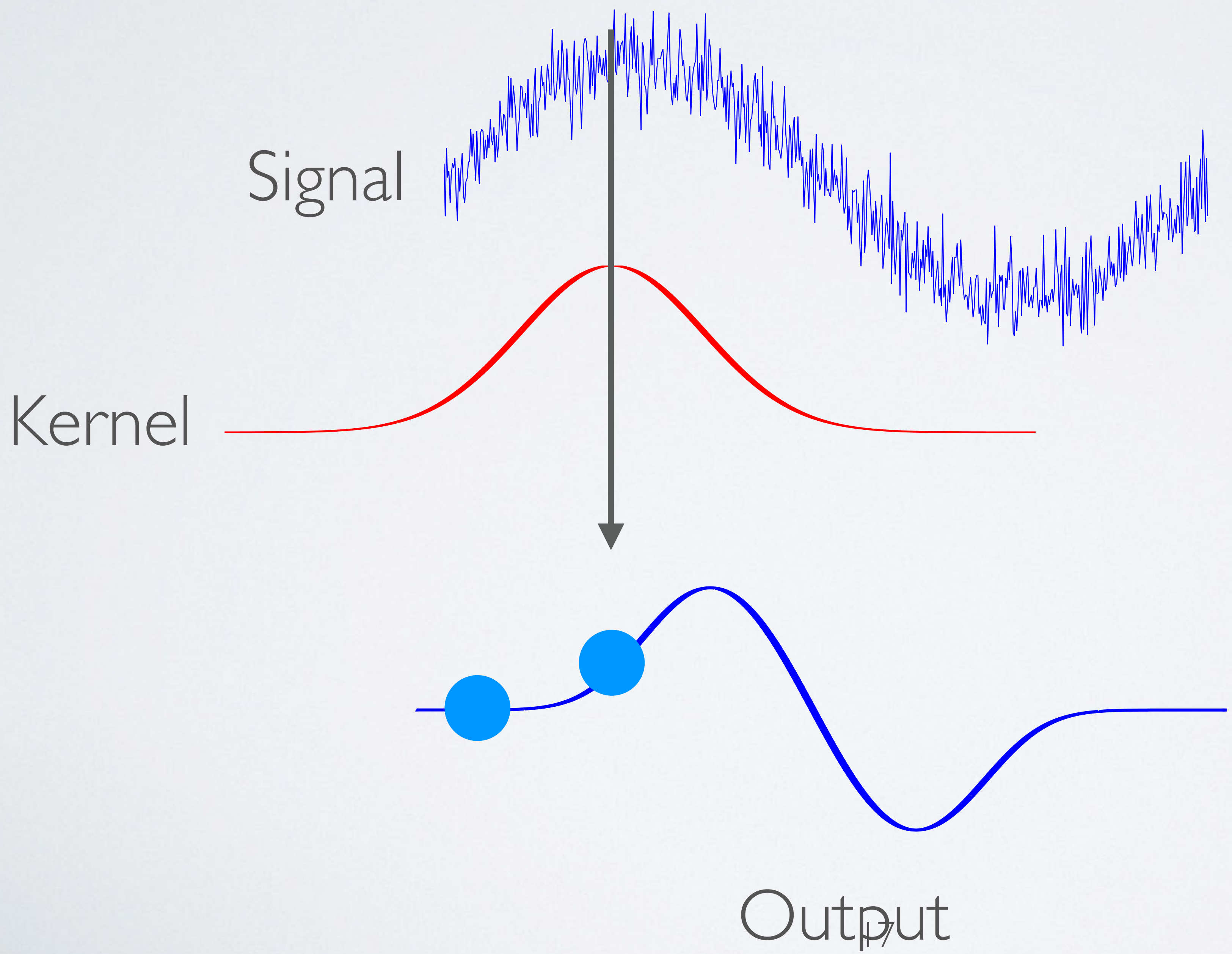
Used to "smooth" a noisy signal



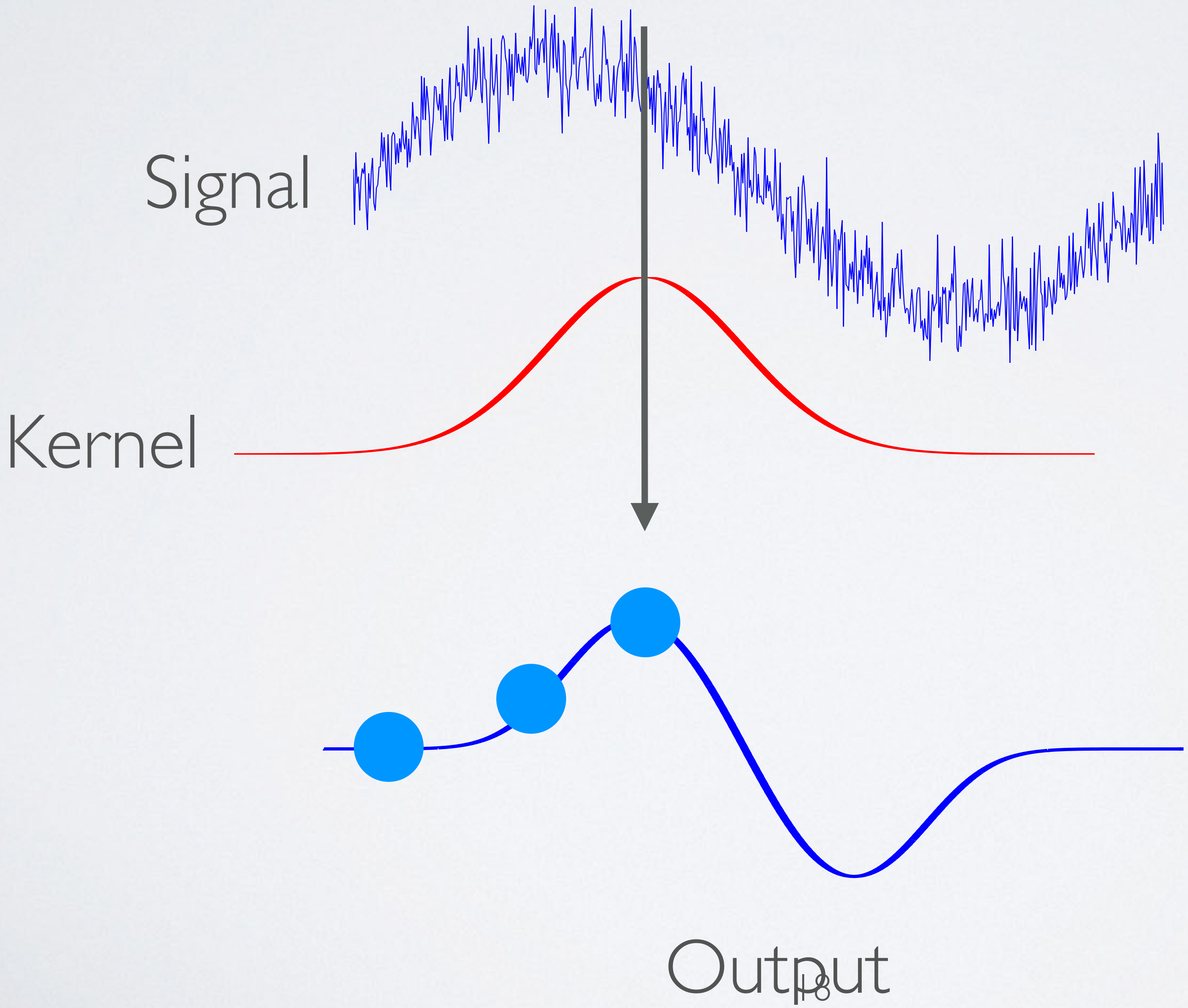
GAUSSIAN FILTER



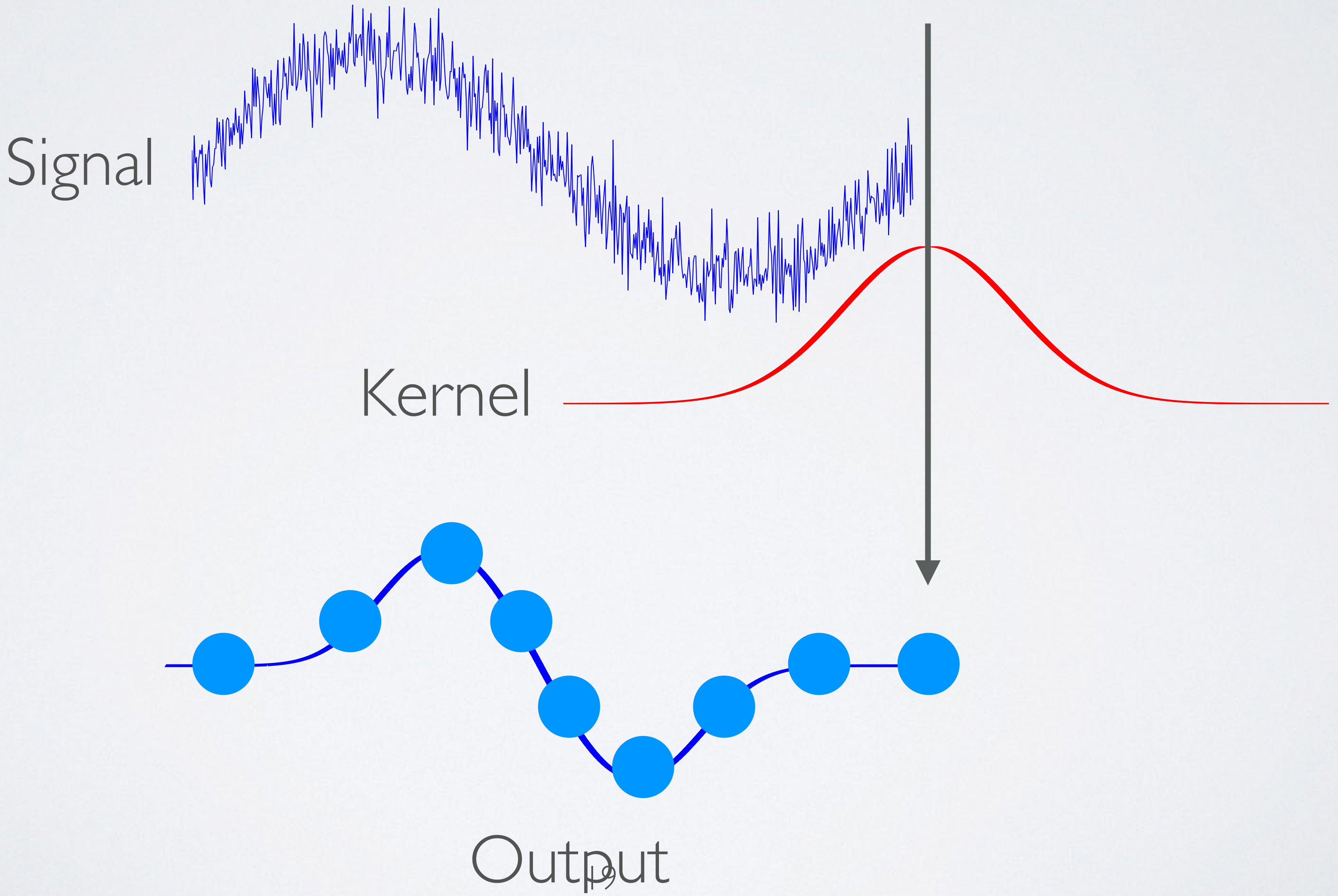
GAUSSIAN FILTER



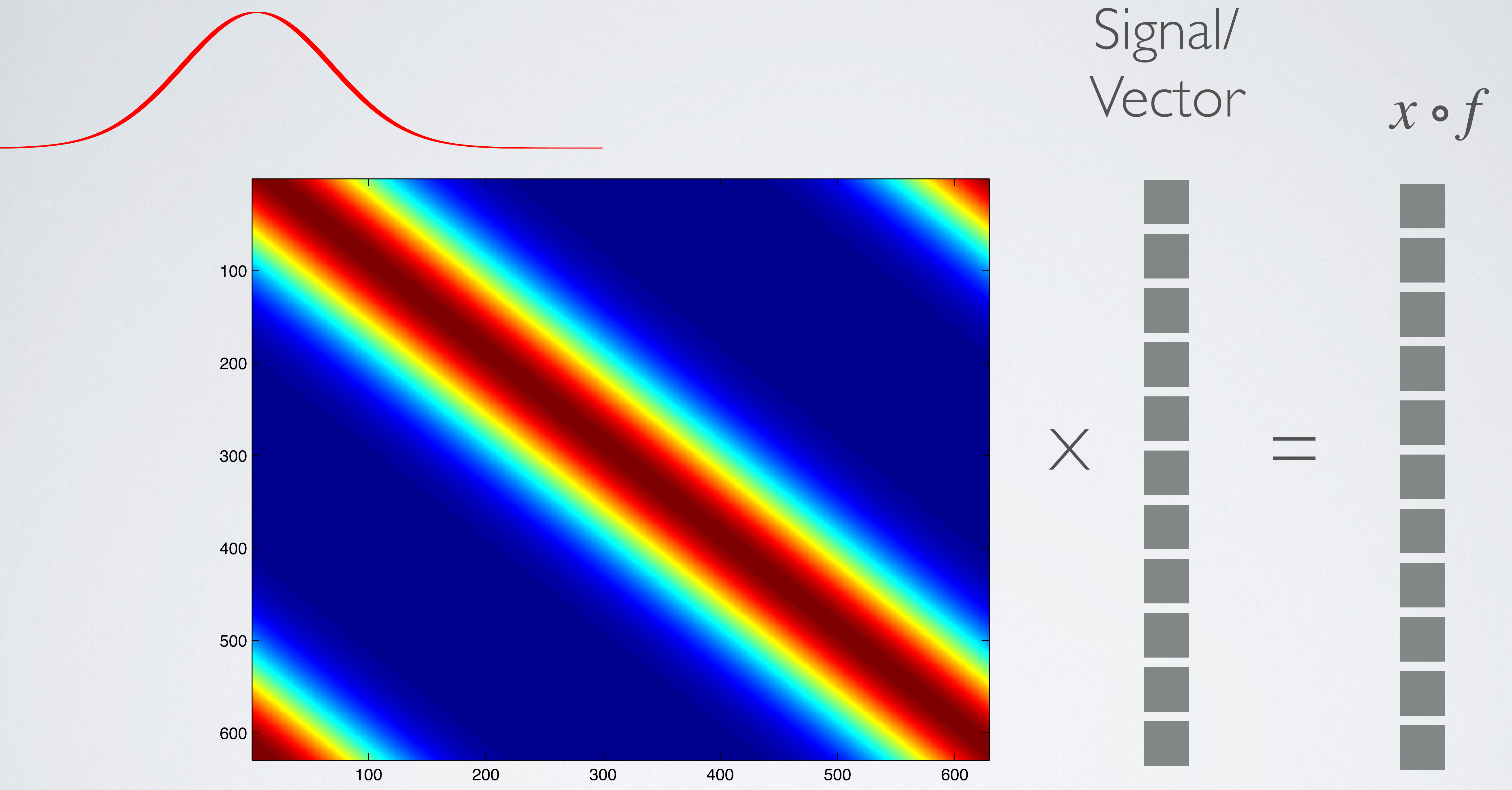
GAUSSIAN FILTER



GAUSSIAN FILTER



CONVOLUTION MATRIX



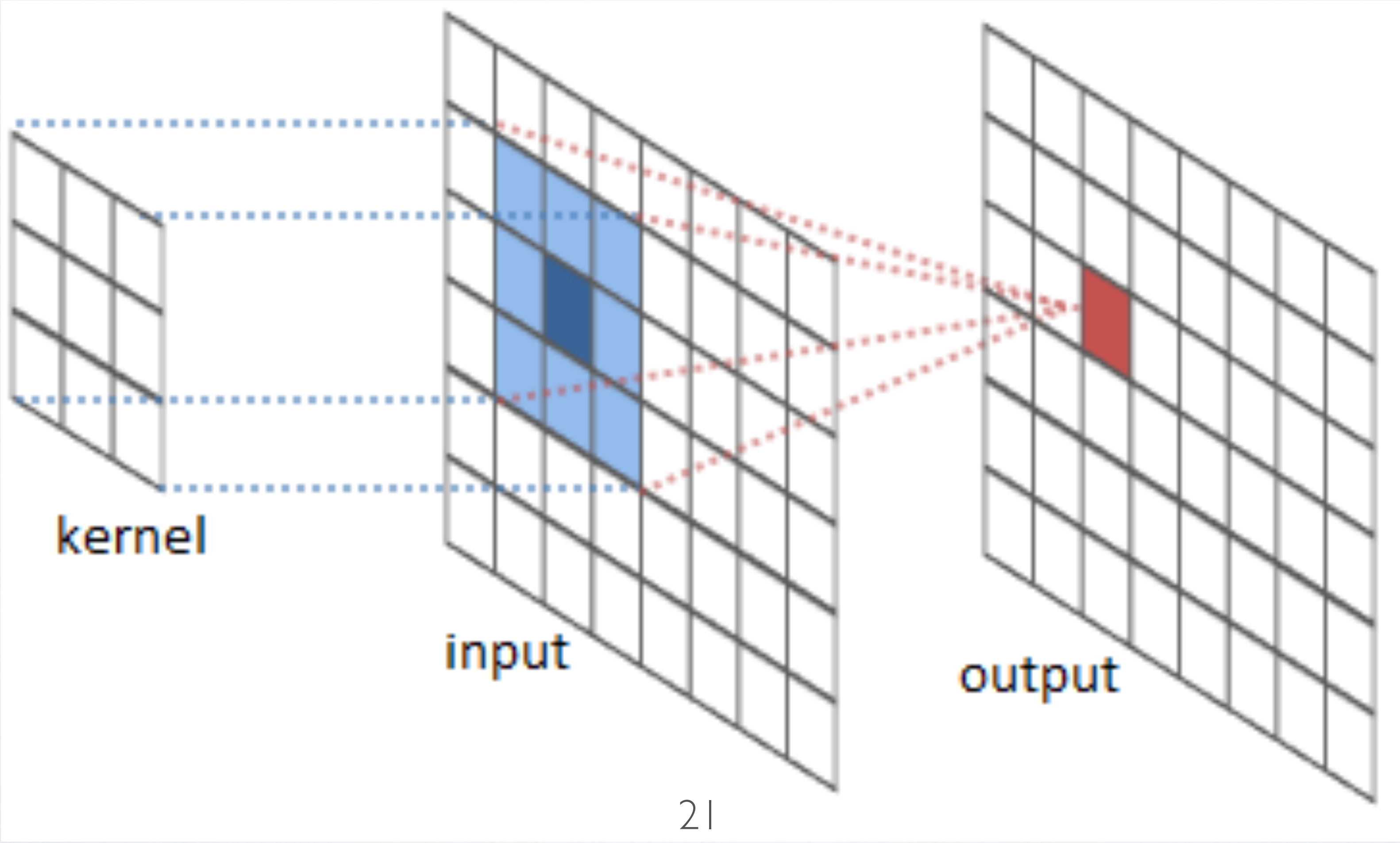
CONVOLUTION

$$f * g(x) = \sum_s f(x - s)g(s) = \sum_s f(x + s)g(-s)$$

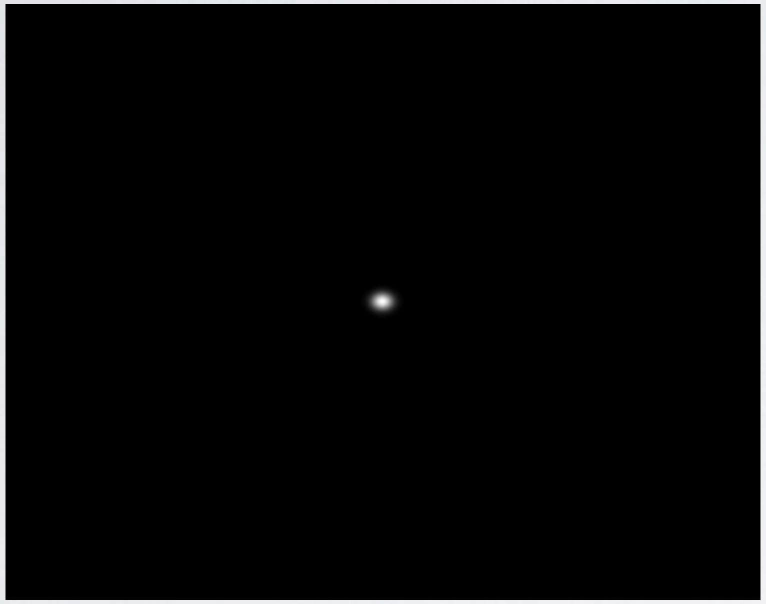
don’t forget

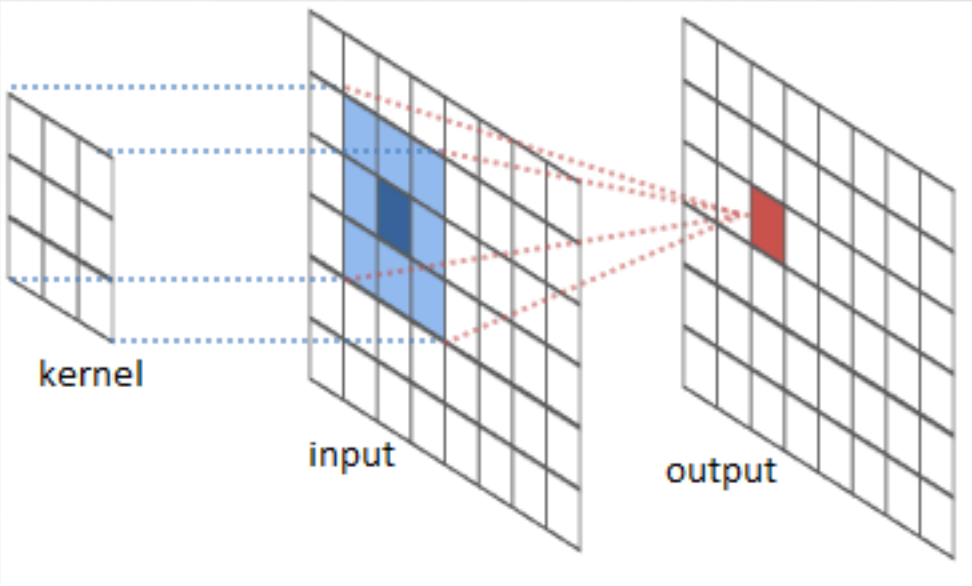
WRONG!!

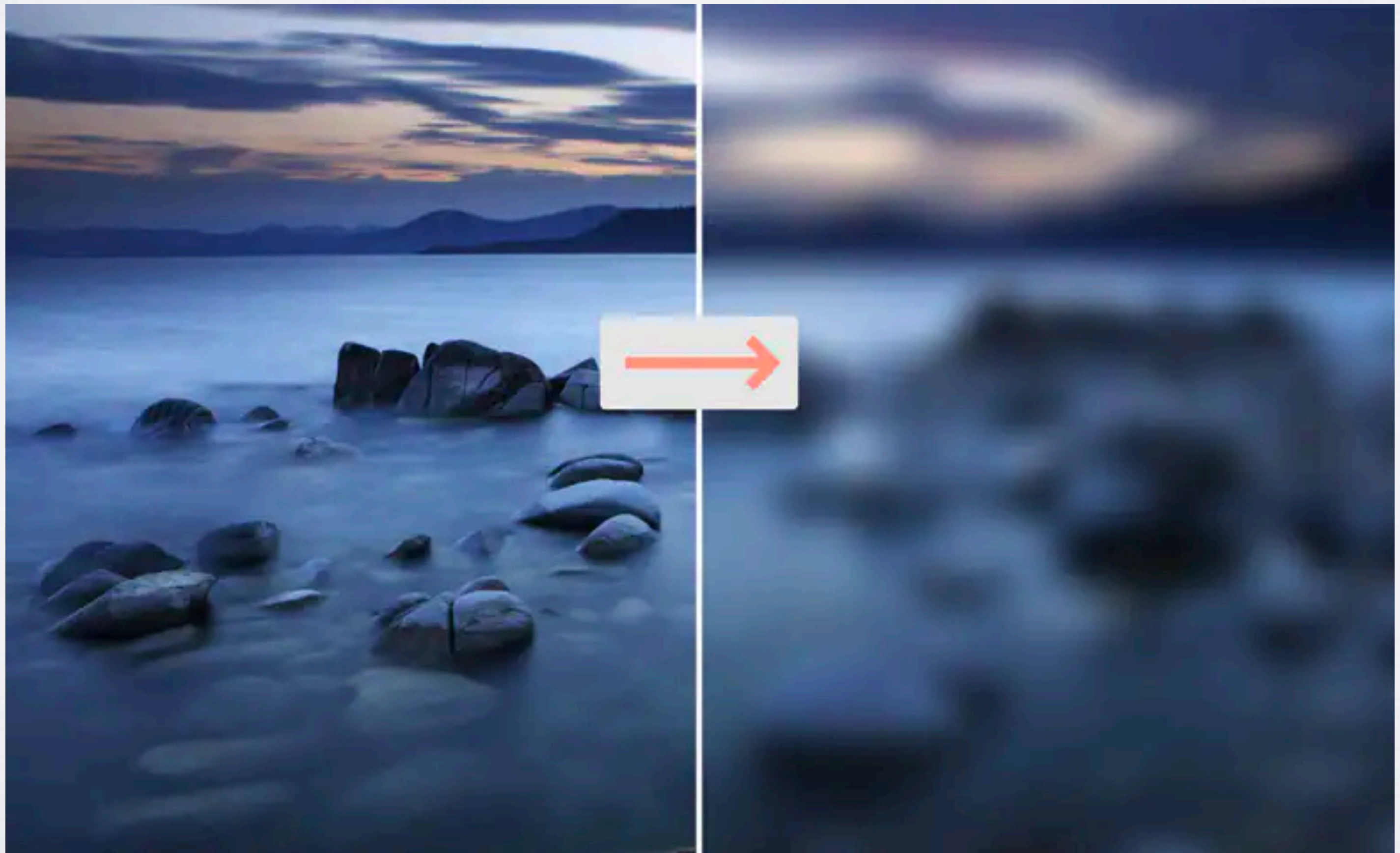
backwards kernel



CONVOLUTION: 2D





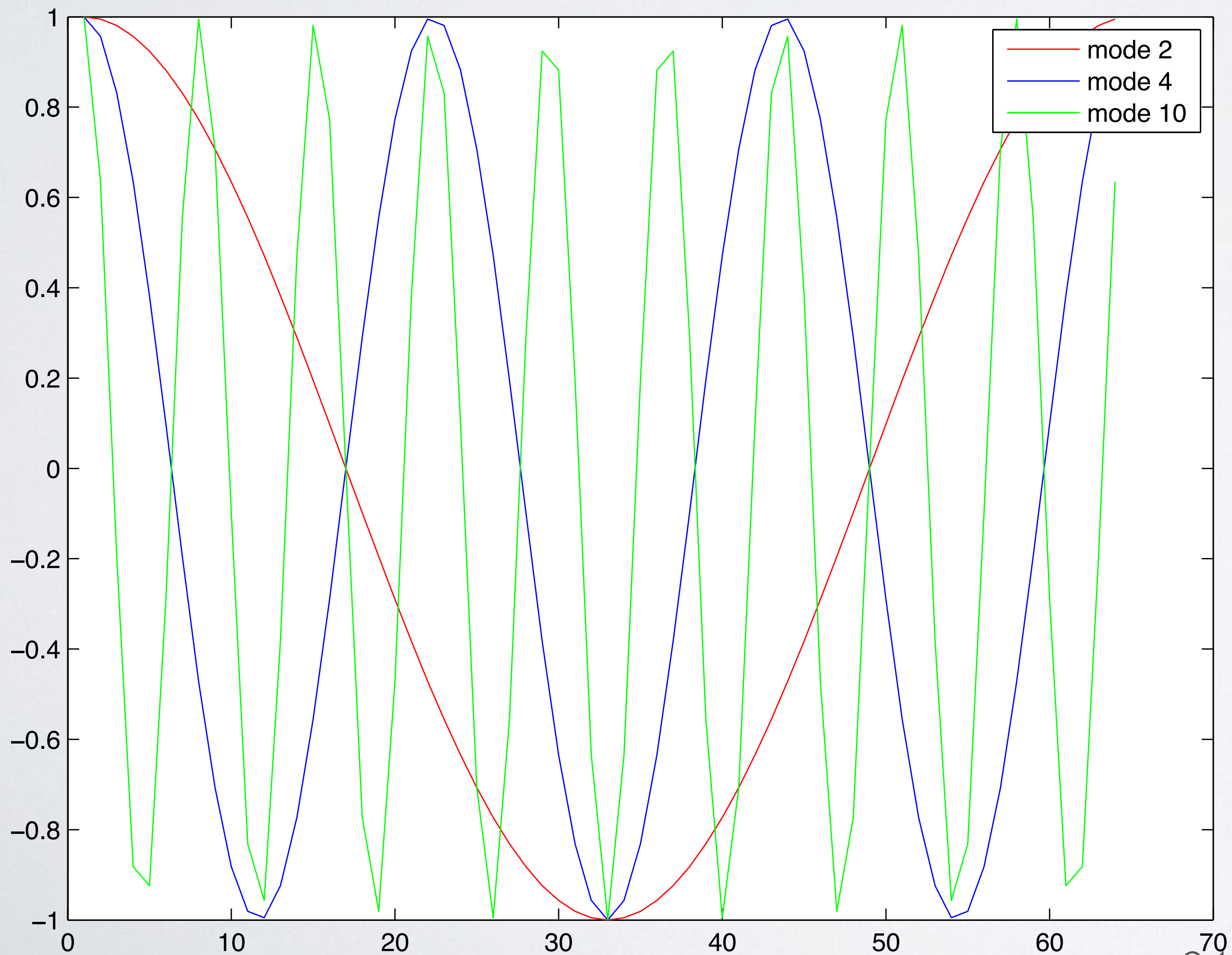


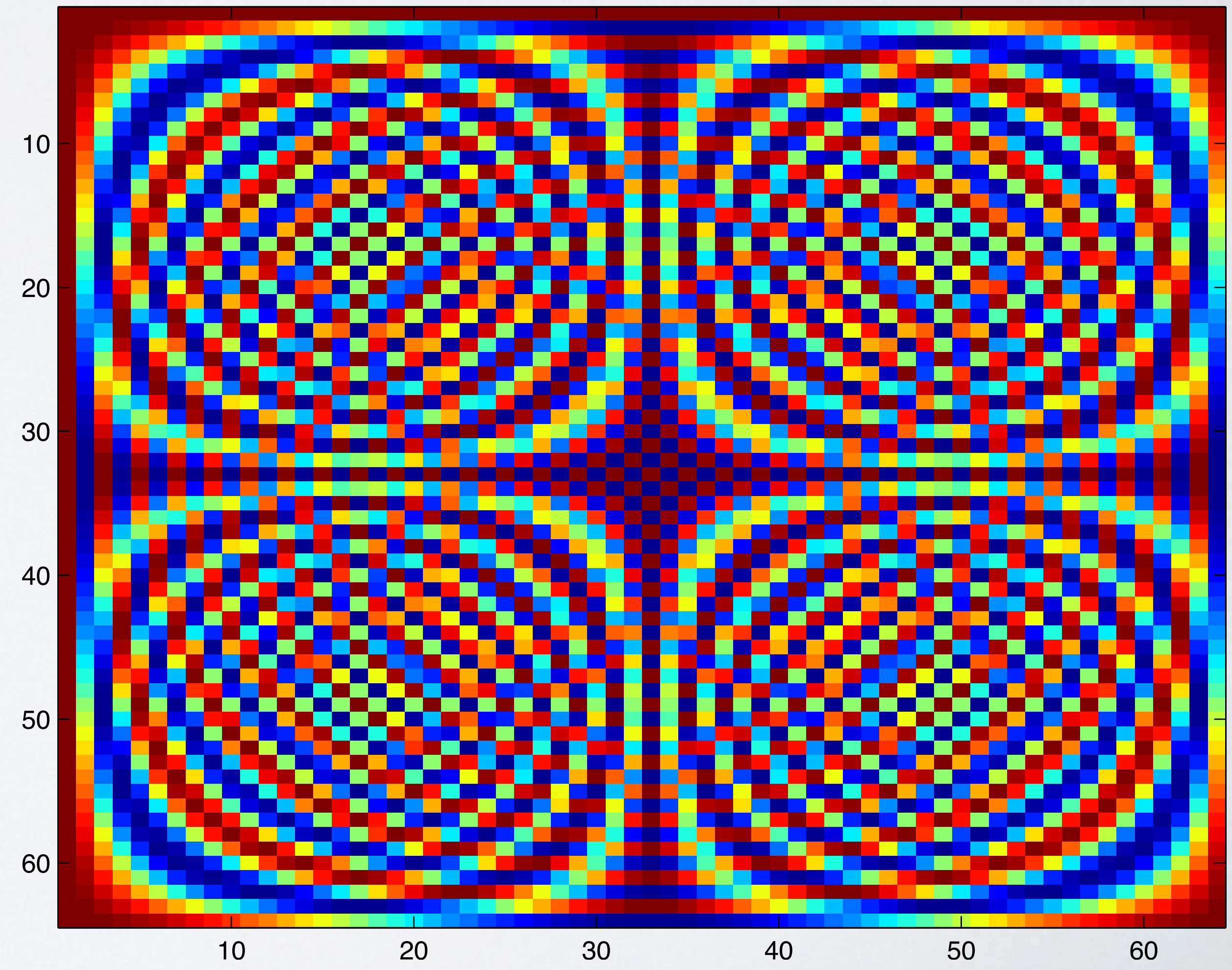
USING THE FFT

FOURIER TRANSFORM

$$\hat{x}_k = \sum_n x_n e^{-i2\pi kn/N}$$

$$\hat{x}_k = < x, \mathcal{F}_k >$$





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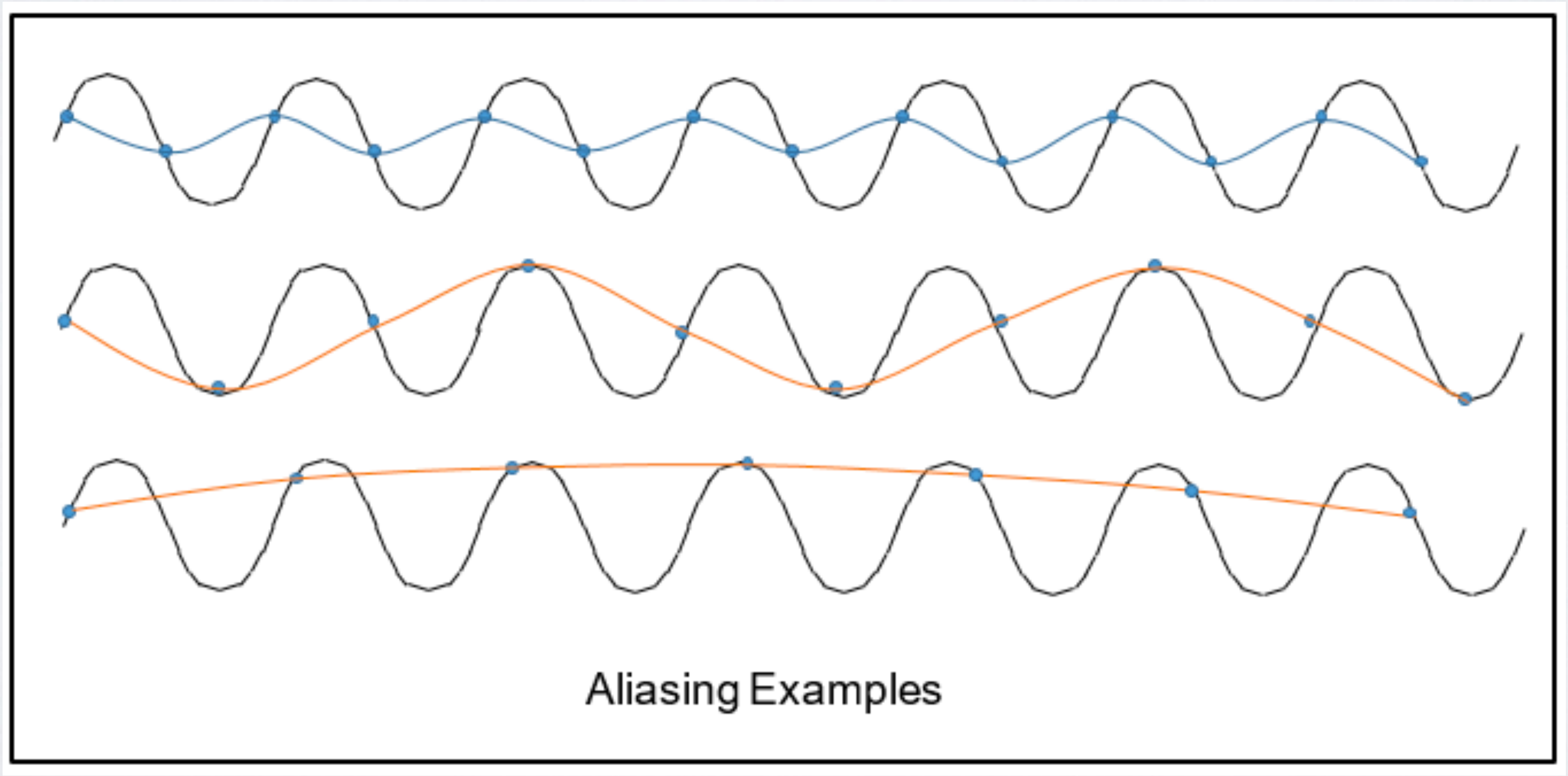


PROPERTIES OF DFT

- Classical FFT is **ORTHOGONAL**

$$\mathcal{F}^H \mathcal{F} = nI$$

- A re-scaling of it is **UNITARY**

$$\mathcal{F}_\perp^H \mathcal{F}_\perp = I$$

- Computable in O(N logN) time (Cooley-Tukey)

$$\mathcal{F}x = \begin{pmatrix} I & B \\ I & -B \end{pmatrix} \begin{pmatrix} \mathcal{F}x_e \\ \mathcal{F}x_o \end{pmatrix}$$

THE CONVOLUTION THEOREM

- Convolution = multiplication in Fourier domain

$$x * y = \mathcal{F}^{-1}(\mathcal{F}x \cdot \mathcal{F}y)$$

what does this mean?

A BETTER WAY TO THINK ABOUT IT

$$x * y = \mathcal{F}^{-1}(\mathcal{F}x \cdot \mathcal{F}y)$$

- FFT **DIAGONALIZES** convolution matrices

Orthogonal

$$Kx = \mathcal{F}^{-1} D \mathcal{F} x$$

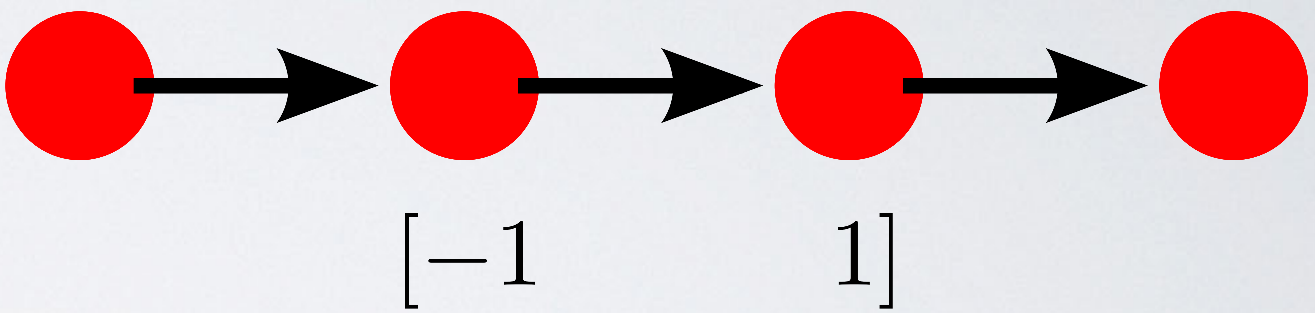
Diagonal

Trivia!

- Is $K_1 K_2 = K_2 K_1$ **?**
- Does $\partial_x \partial_y u = \partial_y \partial_x u$ **?**

HOW DO WE FIGURE OUT D?

- The diagonal matrix is just the FFT of **backward** filter stencil



- Example: derivative
    - Define stencil: [-1 1 0]
    - Convolve with: [0 1 -1]
    - Embed stencil in matrix:

$$s = [1 \; -1 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0]$$

$$Kx = \mathcal{F}^{-1} D \mathcal{F} x$$

- Compute the diagonal

$$D = \mathcal{F} s$$

- Perform convolution using EVD

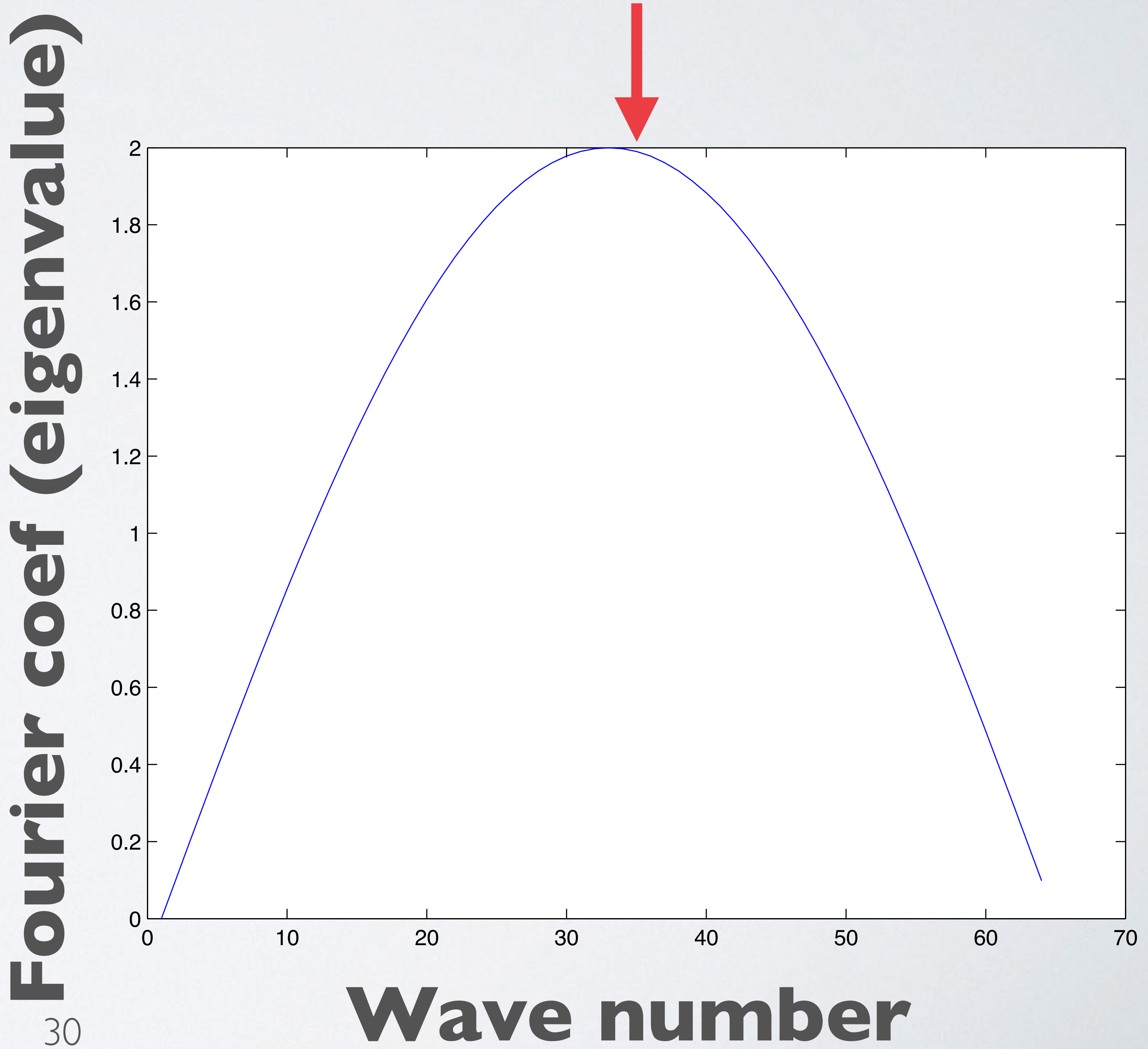
CONDITION NUMBER

$$Kx = \mathcal{F}^{-1} D \mathcal{F} x$$

$$\text{diag}(D) = \mathcal{F}[1, -1, 0, 0, 0, 0]$$

How poorly conditioned is the FFT?

How poorly conditioned is a convolution?



CONDITION NUMBER

$$Kx = \mathcal{F}^{-1} D \mathcal{F} x$$

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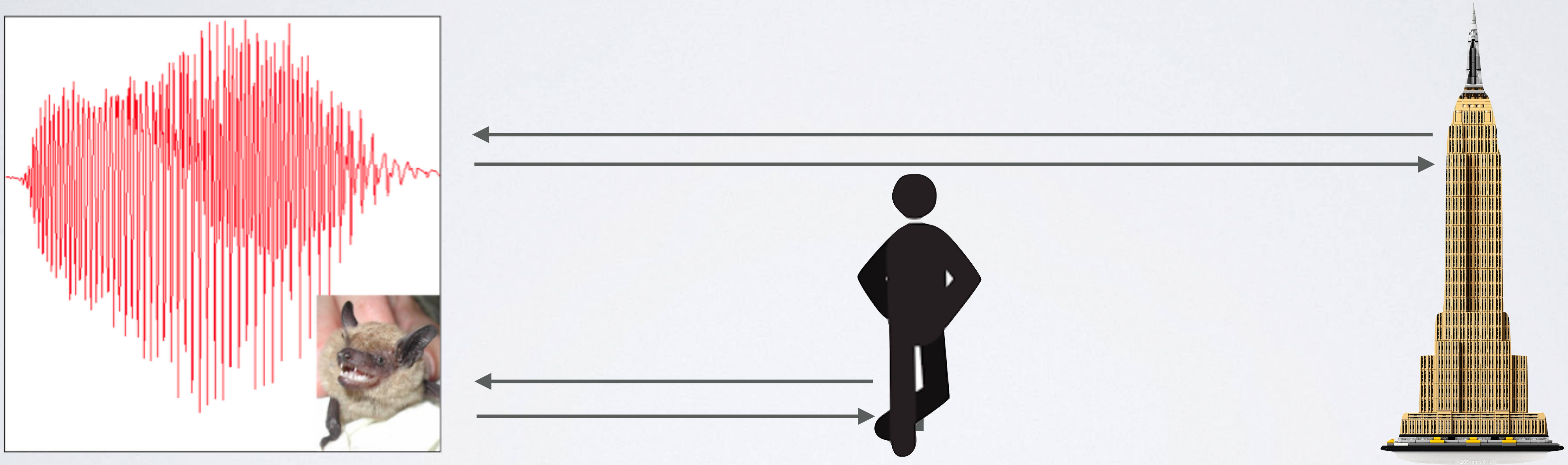
How poorly conditioned is the FFT?

How poorly conditioned is a convolution?

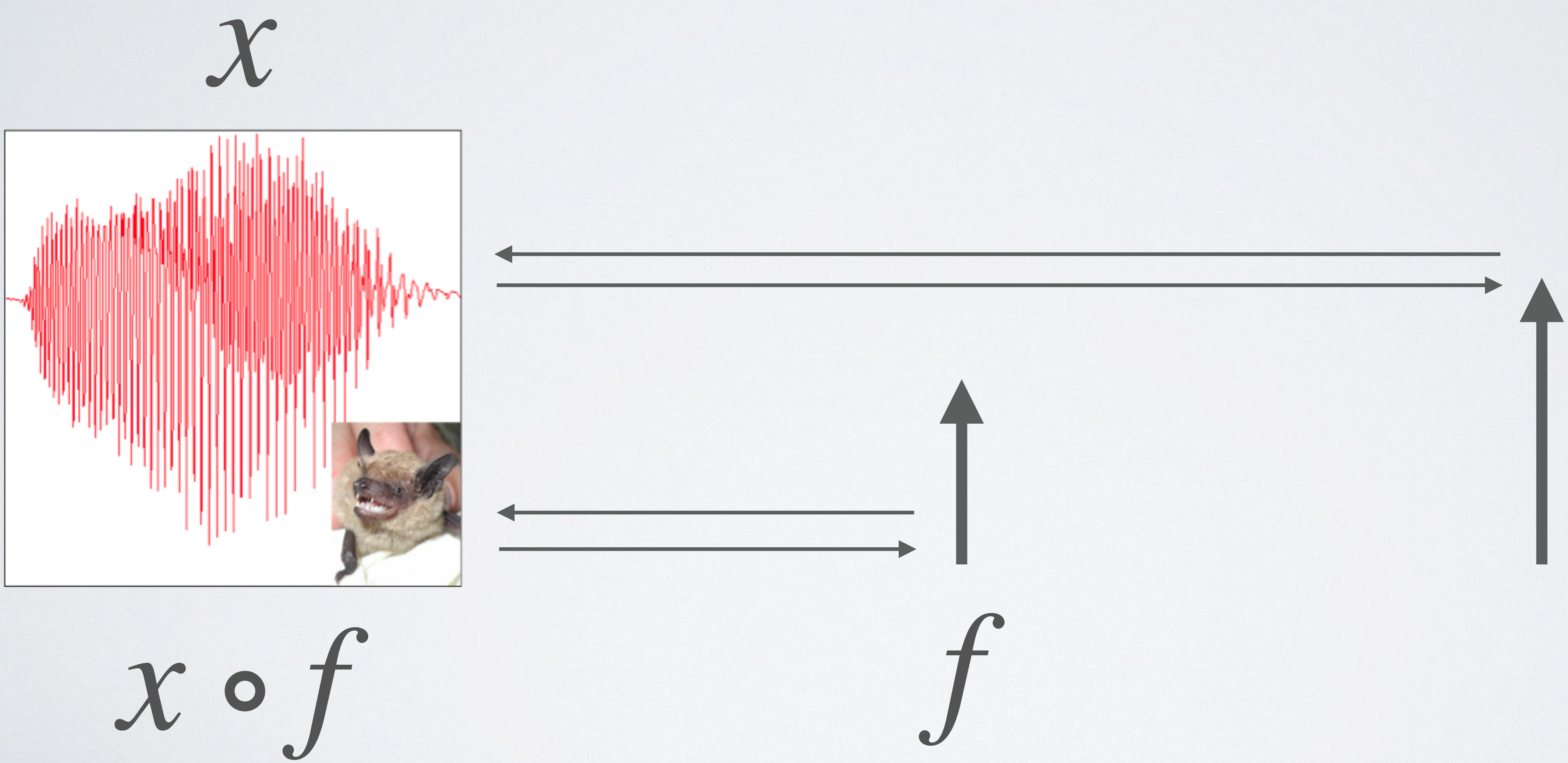
Zero: why? What is the eigenvector

BROWN BAT

x

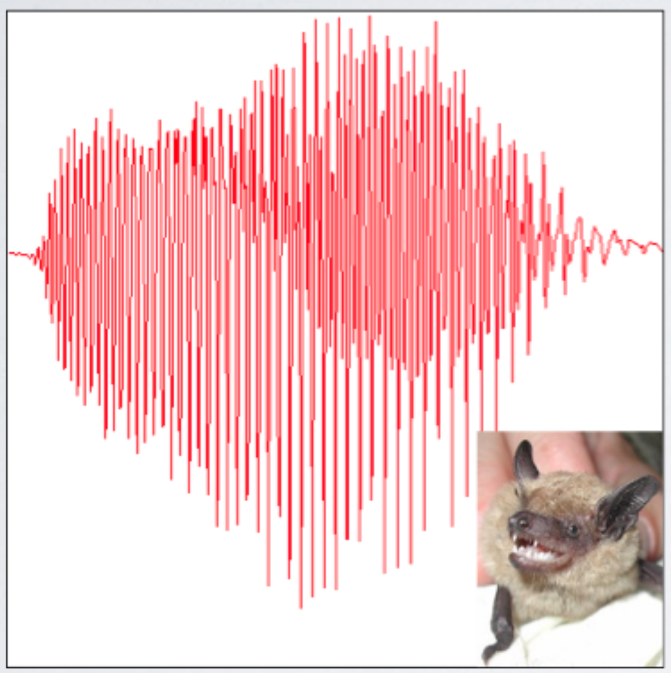


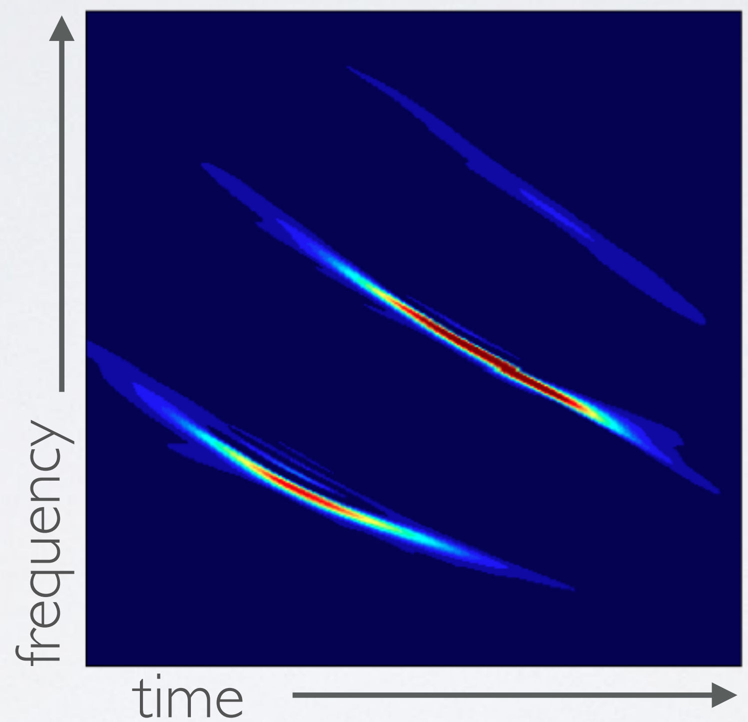
BROWN BAT



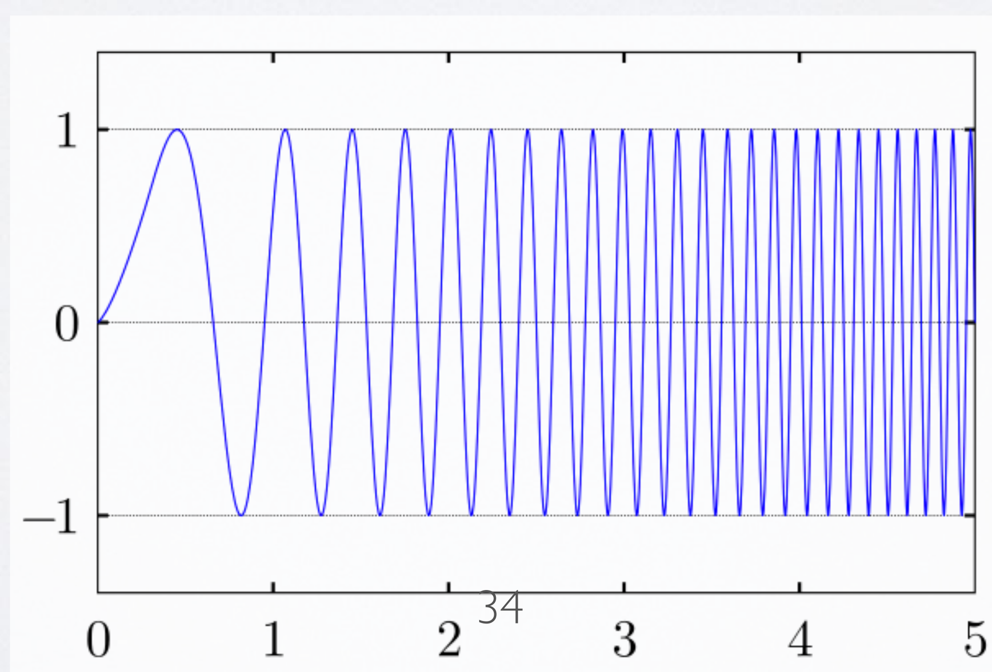
WHY IS THE CHIRP WELL-CONDITIONED?

Echolocation chirp: brown bat





Linear chirp



FFT IN THE WILD

Matlabs DFT is NOT ORTHOGONAL!

WHY?

Convolution with a delta function = identity

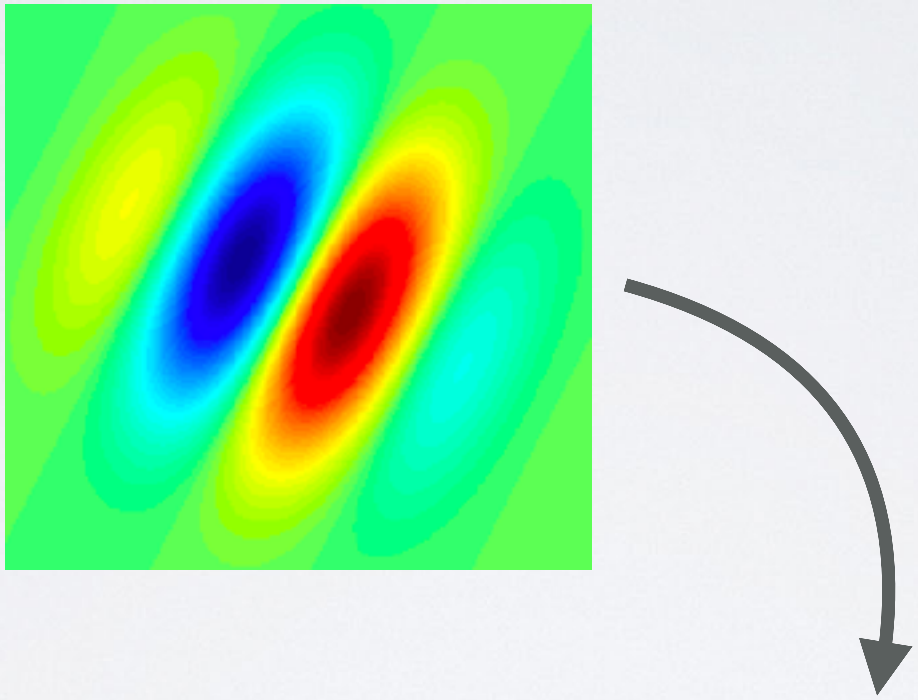
$$\mathcal{F}_\perp = \frac{1}{\sqrt{N}}\mathcal{F} \qquad \mathcal{F}_\perp^{-1} = \sqrt{N}\mathcal{F}^{-1}$$

FFT still diagonalizes convolutions!

$$\mathcal{F}^{-1}D\mathcal{F} = \sqrt{N}\mathcal{F}^{-1}D\mathcal{F}\frac{1}{\sqrt{N}} = \mathcal{F}_\perp^{-1}D\mathcal{F}_\perp$$

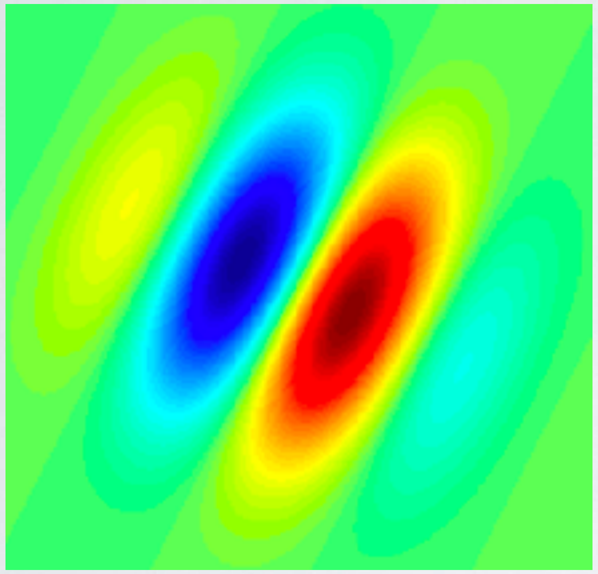
Same D

WHAT'S THE ADJOINT OF A CONV OPERATOR?



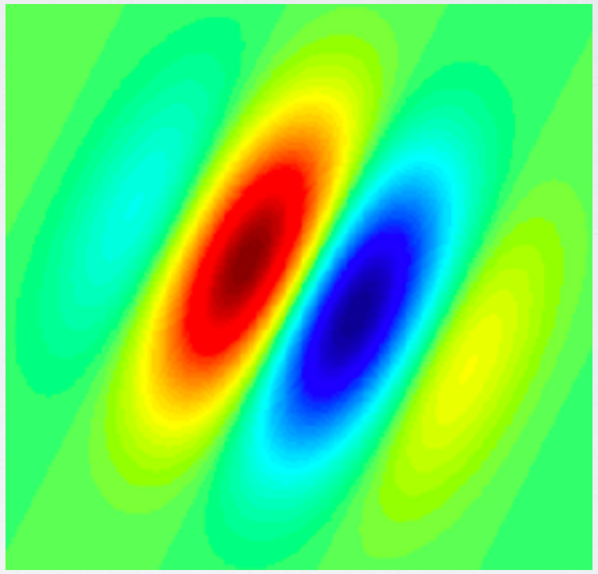
$$Kx = \mathcal{F}^{-1} D \mathcal{F} x$$

WHAT'S THE ADJOINT OF A CONV OPERATOR?



$$Kx = \mathcal{F}^{-1} D \mathcal{F} x$$

Flip!



$$K^H x = \mathcal{F}^{-1} \bar{D} \mathcal{F} x$$