MATH 130: CALCULUS FOR THE LIFE SCIENCES I

Quiz 8

1. Find the locations of all absolute extrema for the function: \( f(x) = \frac{x}{x^2+2} \) with domain: [0,4] (4 points)

2. Find the nonnegative numbers x and y that maximize \( xy^2 \) and satisfy the \( x+y=45 \). Give the optimum value of \( xy^2 \). (6 points)
SOLUTIONS OF QUIZ 8

1. 
\[ f(0) = \frac{0}{0+2} = 0 \]
\[ f(4) = \frac{4}{4^2 + 2} = \frac{4}{18} = \frac{2}{9} \]
\[ f'(x) = \frac{x'(x^2 + 2) - x(x^2 + 2)' - x(2x)}{(x^2 + 2)^2} = \frac{2 - x^2}{(x^2 + 2)^2} \]

\[ f'(x) = 0 \Rightarrow 2 - x^2 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \pm \sqrt{2} \]

But \(-\sqrt{2}\) is out of the domain \([0,4]\) so the only critical number is \(\sqrt{2}\).

\[ f(\sqrt{2}) = \frac{\sqrt{2}}{\sqrt{2}^2 + 2} = \frac{\sqrt{2}}{2 + 2} = \frac{\sqrt{2}}{4} \]

But, \(0 < \frac{2}{9} < \frac{\sqrt{2}}{4}\)

So the absolute minimum is 0 and the absolute maximum is \(\frac{\sqrt{2}}{4}\) at the locations 0 and \(\sqrt{2}\) respectively.

2. There are two different ways to solve this exercise with the procedure that we presented in class. The one is to substitute \(x\) and it is the easiest and the other is to substitute \(y\). We only need to use one of the two but we will present both here.

a)

\[ x + y = 45 \Rightarrow x = 45 - y \text{ and } x \geq 0. \text{ So, } 45 - y \geq 0 \Rightarrow 45 \geq y \Rightarrow y \leq 45. \text{ That means that } 0 \leq y \leq 45. \]

Now,

\[ I = xy^2 = (45 - y)y^2 = 45y^2 - y^3. \]
We need to maximize this, so we find the values at the end-points and the critical numbers.

\[ I(0) = 0 \]
\[ I(45) = 0 \]

\[ I'(y) = 90y - 3y^2 = 3y(30 - y), \quad I'(y) = 0 \Rightarrow y = 0 \text{ or } y = 30 \]

We have already found that
\[ I(0) = 0, \text{ so now, } I(30) = 13,500. \] We conclude that the maximum value is 13,500 for \( y = 30 \) and \( x = 45 - y = 15 \).

b)

\[ x + y = 45 \Rightarrow y = 45 - x \text{ and } y \geq 0. \] So, \( 45 - x \geq 0 \Rightarrow 45 \geq x \Rightarrow x \leq 45. \] That means that \( 0 \leq x \leq 45 \).

Now,
\[ I = xy^2 = x(45 - x)^2 = x(2025 - 90x + x^2) = 2025x - 90x^2 + x^3. \]

We need to maximize this, so we find the values at the end-points and the critical numbers.

\[ I(0) = 0 \]
\[ I(45) = 0 \]

\[ I'(x) = 2025 - 180x + 3x^2 = 3(675 - 60x + x^2) = 3(x - 45)(x - 15), \]
\[ I'(x) = 0 \Rightarrow x = 45 \text{ or } x = 15 \]

We have already found that
\[ I(45) = 0, \text{ so now, } I(15) = 13,500. \] We conclude that the maximum value is 13,500 for \( x = 15 \) and \( y = 45 - x = 30 \).