Epsilon-net method for optimizations over separable states

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Warm Up

Given $\mathbf{H} \in \mathrm{Herm}(\mathcal{X})$ as input. Consider

$$\max / \min \langle \mathbf{H}, \rho \rangle \ \textbf{subject to} \ \rho \in \mathrm{D}(\mathcal{X})$$

- Capture some physical meaning, such as finding the *ground energy/states* when H is a Hamiltonian.
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Given $\mathbf{H} \in \mathrm{Herm}\,(\mathcal{X} \otimes \mathcal{Y})$ as input. Consider

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$$\lambda - \delta \leq \mathit{OptSep}(\mathbf{H}) \leq \lambda + \delta$$

- NP-hard even to solve the problem with $\delta = O(\frac{1}{\text{poly}(d)})$.
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- δ is larger than inverse-polynomial.
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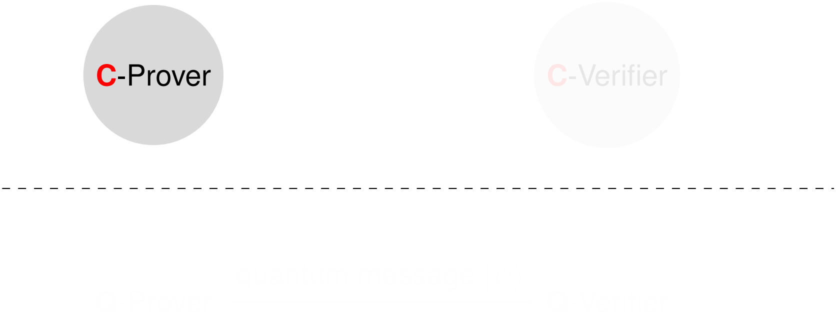

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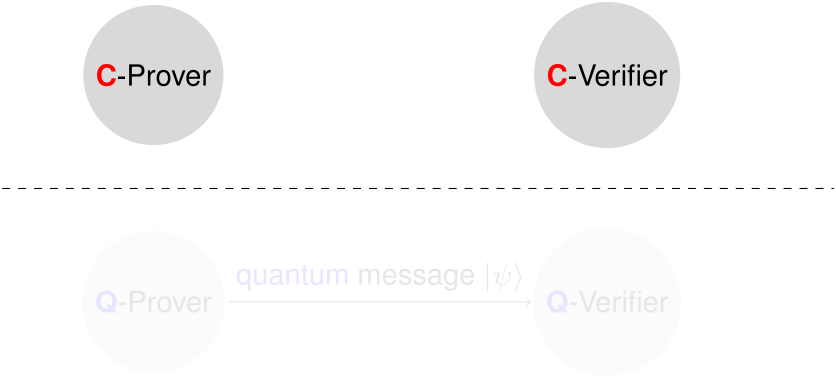

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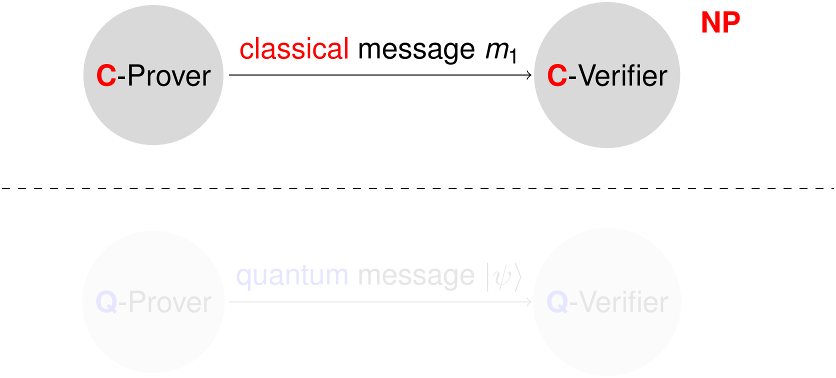

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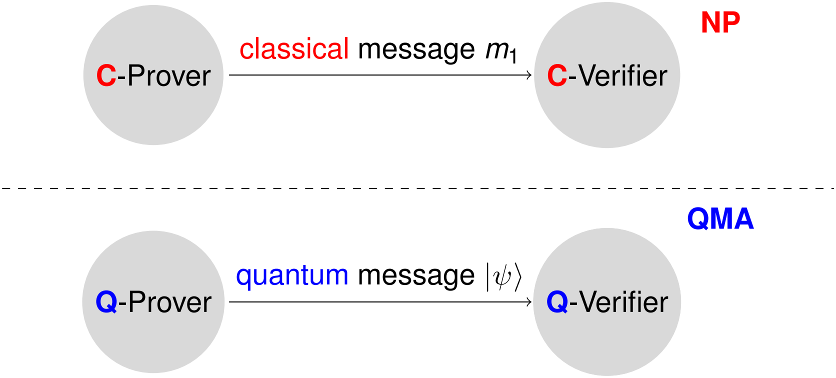

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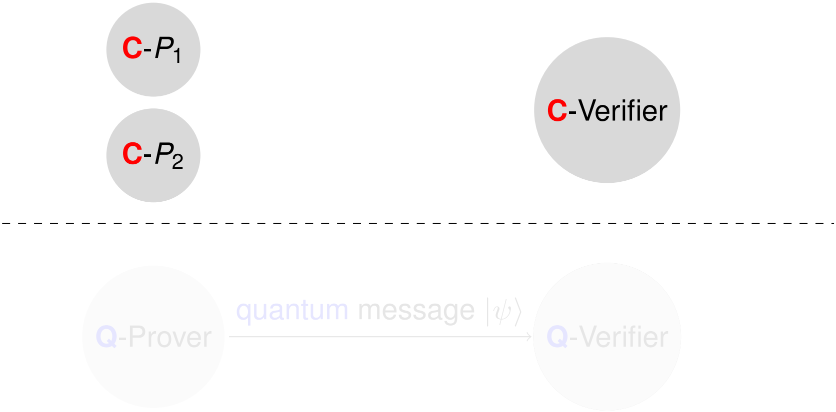

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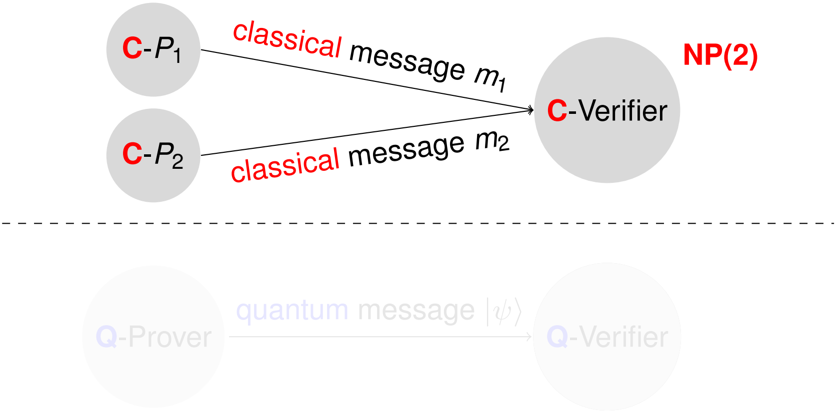

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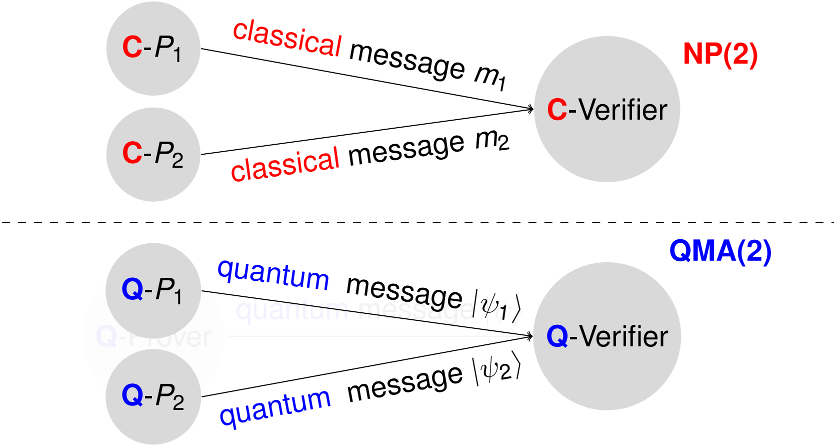

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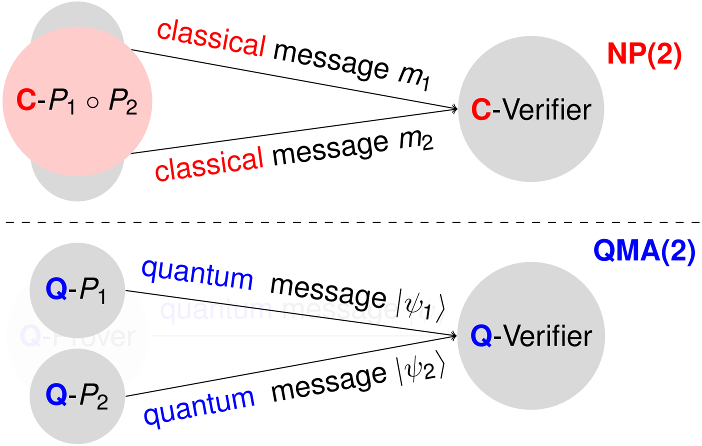

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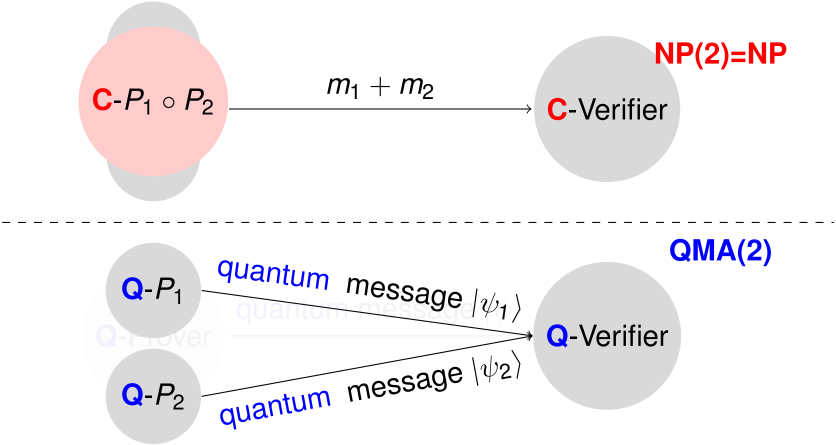

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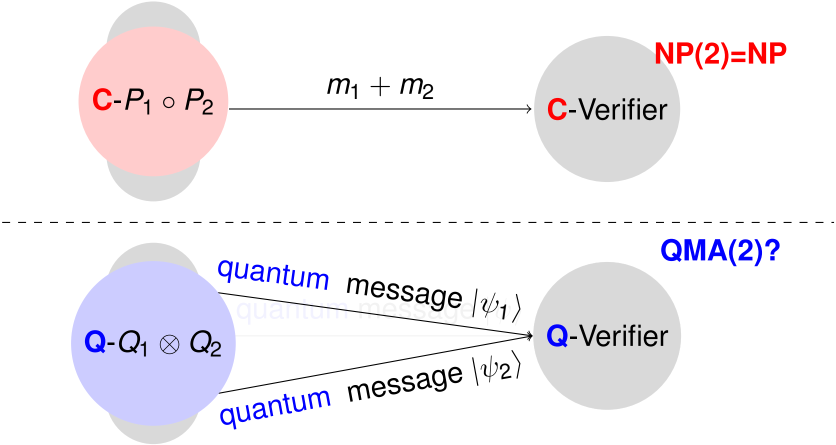

History about QMA(2)

- QMA(2) was firstly studied in **[KMY01, KMY03]**.
- Recently, a surprising result shows NP $\subseteq$ QMA(2)$_{\log}$ **[BT09]**. Compare with QMA$_{\log}$ $=$ BQP **[MW05]**.
- Various improvements of the above result have been found **[Bei10, ABD+09, CF11, GNN11, CD10, ...]**.

However, only the trivial upper bound NEXP is known for QMA(2) and it is widely believed that QMA(2)$\subsetneq$NEXP.

- Variants of QMA(2) have been studied, such as Bell-QMA, LOCC-QMA, which collapse to QMA [Bra08, ABD+09, BCY11].
- QMA(2)=QMA(poly) [HM10].

Better upper bounds, such as EXP, PSPACE, are expected.

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Formulation of QMA(2)

Definition (QMA(2))

A language $\mathcal{L}$ is in QMA(2) if there exists a polynomial-time generated two-outcome POVM measurement $\{Q_x^{\text{acc}}, I - Q_x^{\text{acc}}\}$ for any input x such that,

- If $x \in \mathcal{L}, \exists \left|\psi_1\right\rangle, \left|\psi_2\right\rangle, \langle Q_x^{\text{acc}}, |\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2|\rangle \geq \frac{2}{3}$.
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Roughly it amounts to computing OptSep($\mathbf{Q}_\mathbf{x}^{\text{acc}}$), except

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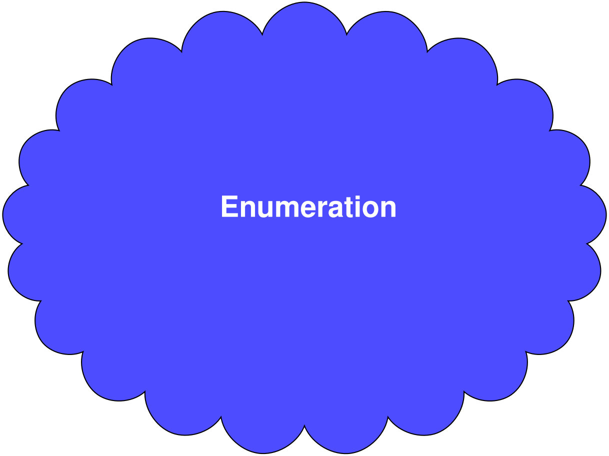

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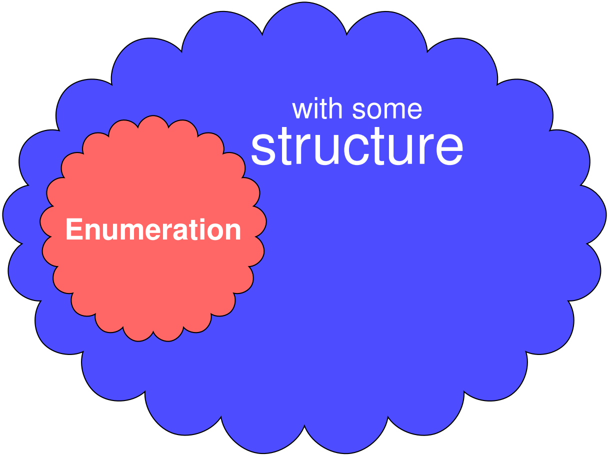

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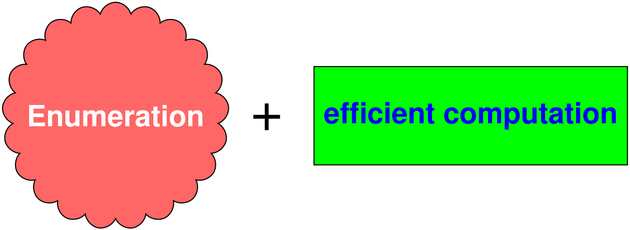

Result based on the DECOMPOSABILITY of H

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We call **H** is $(M, \vec{w})$-decomposable if $H = \sum_{i=1}^{M} H_i^1 \otimes H_i^2$ where $\|H_i^1\| \leq w_1, \|H_i^2\| \leq w_2$.

Intuition: the smaller M is , the more "local" **H** is and the less connection there is between the two parties.

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- The variant QMA(2)[poly(n),$O(\log(n))$]$\subseteq$ PSPACE.
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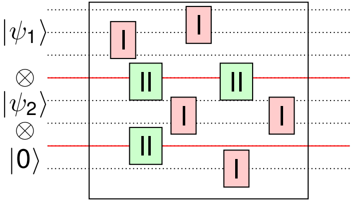

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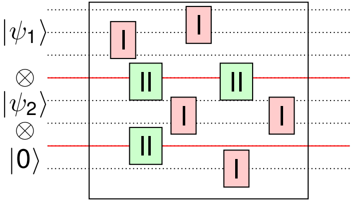

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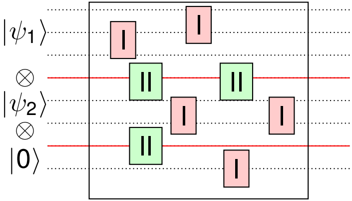

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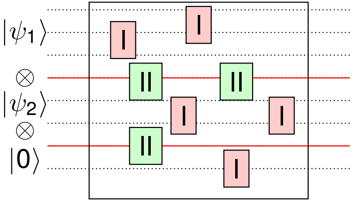

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Assume $H = \sum_{i=1}^{M} H_i^1 \otimes H_i^2$, then we have

$$\max \langle H, \rho_1 \otimes \rho_2 \rangle = \max \langle H_1^1, \rho_1 \rangle \langle H_1^2, \rho_2 \rangle + \cdots + \langle H_M^1, \rho_1 \rangle \langle \mathcal{H}_M^2, \rho_2 \rangle$$

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- Once the ○ values are fixed, the optimization over ○ becomes *efficiently* solvable.
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- Enumerate raw values of $\bigcirc$ terms from a **bounded** set according to $\vec{w}$.
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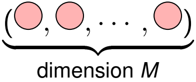

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When $\mathbf{H} \geq 0$, *OptSep*$(\mathbf{H})$ *can be approximated with additive error* δ *in time* $\exp(O(\log(d) + \delta^{-2}\|\mathbf{H}\|_F^2 \ln(\|\mathbf{H}\|_F/\delta)))$.

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Also let $|u\rangle |v\rangle = \sum_t \beta_t |\psi_t\rangle$.

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standard ϵ-net → $\max_{\alpha\in\mathbf{B}(\mathbb{C}^{|\Gamma_\epsilon|})}$

Schmidt decomposition → $\max_{|u\rangle|v\rangle}$

Conclusion

In this talk, we provide two algorithms based on the following structures of **H**.

- The decomposability of **H**.
- The eigenspace of high eigenvalues of **H**.

Open Problems:

- Algorithm or Hardness for larger δ.
- Upper bound for QMA(2).

Question And Answer

Thank you!