Quantum algorithms for semidefinite programs and convex optimization

Xiaodi Wu

QuICS, University of Maryland

UNIVERSITY OF MARYLAND

JOINT CENTER FOR QUANTUM INFORMATION AND COMPUTER SCIENCE

Outline

Landscape of Quantum Advantage in Optimization

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- **Heuristic**: adiabatic, QAOA for near-term devices,
- **Provable**: our focus, by quantizing classical algorithms.

Summary of Results

- **Convex Optimization** (arXiv: 1809.01731): a quantum algorithm using $\tilde{O}(n)$ queries to the *evaluation* and the *membership* oracles, whereas the best known classical algorithms makes $O(n^2)$ such queries. (*independent work*: arXiv:1809.00643)

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- **Quantum SDP solvers** (arXiv: 1710.02581v2): a quantum algorithm solves n-dimensional semidefinite programs with m constraints, sparsity s and error ϵ in time $\tilde{O}((\sqrt{m}+\sqrt{n})s^2(Rr/\epsilon)^8)$ where R, r are bounds on the primal/dual solutions.

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Yes, we do have accompanying **lower bounds**. Will show!

A generic iterative optimization algorithm

A typical classical iterative algorithm:

- Assume a feasible set P. Want to optimize $f(x)$ s.t. $x \in P$.
- A generic iterative algorithm with T iterations:
- $x_1 \to x_2 \to \cdots \to x_T$. Cost for each step: (1) store x_i; (2) determine x_i based on $x_{i-1}, \cdots, x_1$, P, $f(x)$.

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How quantum potentially speeds up this procedure?

- Reduce the cost for each step. Make it quantum and/or store x_is quantumly. However, this could **complicate** the determination of next x_is.
- Not clear how to reduce the number of iterations T.

Outline

Convex optimization

Convex optimization is a central topic in computer science with applications in:

- Machine learning: training a model is equivalent to optimizing a loss function.
- Algorithm design: LP/SDP-relaxation, such as various graph algorithms (vertex cover, max cut,...)
- ......

Classically, it is a major class of optimization problems that has polynomial time algorithms.

Convex optimization

In general, convex optimization has the following form:

$$\min f(x) \quad \text{s.t. } x \in \mathcal{C},$$

where $\mathcal{C} \subseteq \mathbb{R}^n$ is promised to be a convex body and $f\colon \mathbb{R}^n \to \mathbb{R}$ is promised to be a convex function.

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It is common to be provided with two oracles:

- *membership oracle*: input an $x \in \mathbb{R}^n$, tell whether $x \in \mathcal{C}$;
- *evaluation oracle*: input an $x \in \mathcal{C}$, output $f(x)$.

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Given a parameter $\epsilon > 0$ for accuracy, the goal is to output an $\tilde{x} \in \mathcal{C}$ such that

$$f(\tilde{x}) \leq \min_{x \in \mathcal{C}} f(x) + \epsilon.$$

Convex optimization

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Quantumly, we are promised to have unitaries $O_{\mathcal{C}}$ and O_f s.t.

- for any $x \in \mathbb{R}^n$, $O_{\mathcal{C}}|x\rangle|0\rangle = |x\rangle|I_{\mathcal{C}}(x)\rangle$, where $I_{\mathcal{C}}(x) = 1$ if $x \in \mathcal{C}$ and $I_{\mathcal{C}}(x) = 0$ if $x \notin \mathcal{C}$;
- for any $x \in \mathcal{C}$, $O_f|x\rangle|0\rangle = |x\rangle|f(x)\rangle$.

Convex optimization

Main result. Convex optimization takes

- $\tilde{O}(n)$ and $\Omega(\sqrt{n})$ quantum queries to $O_{\mathcal{C}}$;
- $\tilde{O}(n)$ and $\tilde{\Omega}(\sqrt{n})$ quantum queries to O_f.

Furthermore, the quantum algorithm also uses $\tilde{O}(n^3)$ additional time.

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As a result, we obtain:

- The first nontrivial quantum upper bound on general convex optimization.
- Impossibility of generic exponential quantum speedup of convex optimization! The speedup is at most polynomial.

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Semidefinite programming (SDP)

Given m real numbers $a_1, \ldots, a_m \in \mathbb{R}$, s-sparse $n \times n$ Hermitian matrices $A_1, \ldots, A_m, C$, the SDP is defined as

$$\begin{aligned} \max \quad & \mathrm{tr}[CX] \\ \text{s.t.} \quad & \mathrm{tr}[A_i X] \leq a_i \quad \forall\, i \in [m]; \\ & X \succeq 0. \end{aligned}$$

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SDPs can be solved in polynomial time. Classical *state-of-the-art* algorithms include:

- Cutting-plane method: $\tilde{O}(m(m^2 + n^{2.374} + mns)\operatorname{poly}\log(Rr/\epsilon))$.
- Matrix multiplicative weight: $\tilde{O}(mns(Rr/\epsilon)^7)$.

Quantum algorithms for SDPs

Brandão and Svore gave a quantum algorithm with complexity $\tilde{O}(\sqrt{mn}s^2(Rr/\epsilon)^{32})$, a quadratic speed-up in m, n, (later improved to $\tilde{O}(\sqrt{mn}s^2(Rr/\epsilon)^8)$, based on the **Matrix Multiplicative Weight Update** method.

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Input model

An oracle that takes input $j \in [m+1]$, $k \in [n]$, $l \in [s]$, and performs the map

$$|j, k, l, 0\rangle \mapsto |j, k, l, (A_j)_{k,s_{jk}(l)}\rangle,$$

where $(A_j)_{k,s_{jk}(l)}$ is the l^{th} nonzero element in the k^{th} row of matrix A_j.

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Theorem

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paper	result
BS17	$\tilde{O}(\sqrt{mn}s^2(Rr/\epsilon)^{32})$
vAGGdW17	$\tilde{O}(\sqrt{mn}s^2(Rr/\epsilon)^8)$
this talk	$\tilde{O}((\sqrt{m} + \sqrt{n})s^2(Rr/\epsilon)^8)$

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The behavior of the algorithm:

- The good: optimal in m, n
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Applications:

- The good: Some machine learning, especially compressed sensing problems have $Rr/\epsilon = O(1)$ (Ex. quantum compressed sensing by Gross et al. 09).
- The bad: The SDP in the Goeman-Williams algorithm for MAX-CUT has $Rr/\epsilon = \Theta(n)$ (and many other algorithmic SDP applications).

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Take-away messages for the upper bound

Convex Optimization

$$\mathsf{MEM} \xrightarrow{\tilde{O}(1)} \mathsf{SEP} \xrightarrow{\tilde{O}(n)} \mathsf{OPT}$$

Poly-log quantum queries suffice to approximate sub-gradients.

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Semidefinite Programs

Intermediate States in Matrix Multiplicative Weight Update method:

$$\rho^{(t)} = \frac{\exp\left[\frac{\epsilon}{4}\sum_{\tau=1}^{t-1} M^{(\tau)}\right]}{\mathrm{Tr}[\exp\left[\frac{\epsilon}{4}\sum_{\tau=1}^{t-1} M^{(\tau)}\right]]} \text{(Gibbs state)}.$$

Faster quantum algorithms to sample Gibbs states.

The lower bound

- **Convex Optimization**: Convex optimization takes
  - $\tilde{O}(n)$ and $\Omega(\sqrt{n})$ quantum queries to $O_{\mathcal{C}}$;
  - $\tilde{O}(n)$ and $\tilde{\Omega}(\sqrt{n})$ quantum queries to O_f.
- **Semidefinite Programs**:
  - Upper bound: $\tilde{O}((\sqrt{m}+\sqrt{n})s^2(Rr/\epsilon)^8)$.
  - Lower bound: $\Omega(\sqrt{m}+\sqrt{n})$.

High-level difficulty:

- (1) continuous domain (vs Boolean oracle query);
- (2) classical lower bounds are not studied comprehensively.

Outline

Open questions!

- Can we close the gap for both membership and evaluation queries? Our upper bounds on both oracles use $\tilde{O}(n)$ queries, whereas the lower bounds are only $\tilde{\Omega}(\sqrt{n})$.
- Can we improve the time complexity of our quantum algorithm? The time complexity $\tilde{O}(n^3)$ of our current quantum algorithm matches that of the classical state-of-the-art algorithm.
- What is the quantum complexity of convex optimization with a first-order oracle (i.e., with direct access to the gradient of the objective function)?
- Concrete applications where quantum algorithms (both for convex optimization and SDPs) can have provable speed-ups?

Thank you!

Q & A